A Theory of General Stress Testing

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Outline

Purpose of Stress Testing: Complement statistical risk measurement

- Statistical risk measurements: What are probs of big losses?
  Stress Testing: Which scenarios lead to big losses?
  Derive risk reducing action.
- Statistical risk measurement: Assume fixed model.
  Stress Testing: Consider alternative risk factor distribution.
  Address model risk.

Requirements on stress scenarios (Basel II)

- Plausibility
- Severity
- Suggestive of risk reducing action

Stress Testing with Point Scenarios

Framework:
Prior risk factor distribution ν on $(\Omega, \mathbb{F})$.
Portfolio loss function L on Ω, measurable.

Stress Testing:
Find worst case scenario and worst case loss
in some set $A \subset \Omega$ of point scenarios:

$$\sup_{\mathbf{r} \in A} L(\mathbf{r}) =: \rho_A(-L).$$

Risk measurement:
$\rho_A(-L)$ is a coherent risk measure.

Systematic Stress Testing with Point Scenarios

- For elliptical risk factor distribution ν, introduce **measure of plausibility** for point scenarios:

$$\mathrm{Maha}(\mathbf{r}) := \sqrt{(\mathbf{r} - \mathbb{E}(\mathbf{r}))^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - \mathbb{E}(\mathbf{r}))},$$

  where Σ is covariance matrix of risk factor distribution ν.

- Choose as scenario set A

$$\mathrm{Ell}_k := \{\mathbf{r} : \mathrm{Maha}(\mathbf{r}) \leq k\},$$

  where k is either determined so as to give the ellipsoid some desired probability mass or independent of the number of risk factors.

Advantages of Systematic Stress Testing with Point Scenarios

- Do not miss plausible but severe scenarios.
- Do not consider scenarios which are too implausible.
- Worst case scenario over Ell_k gives information about portfolio structure and suggests risk reducing action.

Problems of Systematic Stress Testing with Point Scenarios

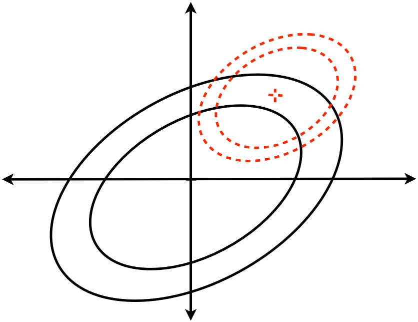
1. Maha does not take into account fatness of tails.
2. How choose scenario set for non-elliptical risk factor distributions ν?
3. $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$ depends on choice of coordinates.
4. $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$ is not law-invariant: Portfolios L_1, L_2 might have the same profit/loss distribution but different $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$.
5. Model risk is not addressed.

Outline

Generalised Scenarios

- Generalised scenario: Probability distribution on $(\Omega, \mathbb{F})$.
- Interpretation 1:
  Point scenarios are generalised scenarios with support concentrated on one point.
- Interpretation 2:
  Risk factor distributions alternative to the prior ν.
  Model risk.

Generalised Scenarios: Interpretation 3



- In multi-period context: Uncertainty is not resolved at intermediate times.
- **Generalised scenario:** Distribution conditional on intermediate realisation.
- **Point scenario:** Intermediate realisation.
- Point scenario equals expectation of generalised scenario for Brownian motion.

Systematic stress testing with generalised scenarios

- Measure of plausibility: relative entropy
  (I-divergence, information gain, Kullback-Leibler distance)

$$I(Q||\nu) := \begin{cases} \int \log \frac{dQ}{d\nu}(\mathbf{r})dQ(\mathbf{r}) & \text{if } Q \ll \nu \\ +\infty & \text{if } Q \not\ll \nu \end{cases}$$

- Scenario set: Instead of ellipsoid take
  Kullback-Leibler sphere in the space of distributions

$$S(\nu, k) := \{Q : I(Q||\nu) \leq k^2/2\}.$$

- Severity of scenarios: Instead of $L(\mathbf{r})$ take $\mathbb{E}_Q(L)$
- Generalised MaxLoss:

$$\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) =: \mathrm{MaxLoss}_k(L)$$

Advantages of Systematic Stress Testing with Generalised Scenarios

1. Relative entropy does take into account fatness of tails of ν.
2. Scenario set is naturally defined for non-elliptical risk factor distributions ν.
3. $\mathrm{MaxLoss}_k$ does not depend on choice of coordinates.
4. $\mathrm{MaxLoss}_k$ is law-invariant: Portfolios L_1, L_2 with the same profit/loss distribution have the same $\mathrm{MaxLoss}_k$.
5. Model risk is addressed:
   Generalised scenarios are alternatives to prior risk factor distribution ν.

Outline

Calculation of generalised MaxLoss

Tool from large deviations theory for solving explicitly the optimisation problem $\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L)$:

$$\Lambda(\theta, L) := \log \left(\int e^{\theta L(\mathbf{r})} d\nu(\mathbf{r}) \right).$$

$\theta_{\max} := \sup\{\theta : \Lambda(\theta) < +\infty\}$

Calculation of generalised MaxLoss

Theorem

The generalised worst case scenario $\overline{Q}$ is the distribution with ν-density

$$\frac{d\overline{Q}}{d\nu}(\mathbf{r}) := \frac{e^{\overline{\theta}L(\mathbf{r})}}{\int e^{\overline{\theta}L(\mathbf{r})}d\nu(\mathbf{r})} = e^{\overline{\theta}L(\mathbf{r})-\Lambda(\overline{\theta})},$$

where $\overline{\theta}$ is the positive solution of

$$\theta\Lambda'(\theta) - \Lambda(\theta) = k^2/2, \tag{1}$$

provided the solution exists. The generalised Maximum Loss achieved in the generalised worst case scenario $\overline{Q}$ is

$$\mathrm{MaxLoss}_k(-L) = \mathbb{E}_{\overline{Q}}(L) = \Lambda'(\overline{\theta}).$$

Law-invariance of generalised MaxLoss

Corollary

$\mathrm{MaxLoss}$ *is a law-invariant risk measure:*
If two portfolios L_1, L_2 *have the same profit-loss distributions,*
$\nu \circ L_1^{-1} = \nu \circ L_2^{-1}$,
then $\mathrm{MaxLoss}_k(-L_1) = \mathrm{MaxLoss}_k(-L_2)$.

MaxLoss from convex conjugate

If Theorem 1 applies, then

$$\Lambda^*(x) = k^2/2 \text{ and } x > \mathbb{E}_\nu(L), \tag{2}$$

where $\Lambda^*(x)$ is the convex conjugate of $\Lambda(\theta)$.

Theorem

(i) *If* $\operatorname{ess\,sup}(L)$ *is finite, and* $k^2/2 \geq -\log(\nu(\{\mathbf{r} : L(\mathbf{r}) = \operatorname{ess\,sup}(L)\}))$, *then* $\mathrm{MaxLoss}_k(-L) = \operatorname{ess\,sup}(L)$.

(ii) *If* $\theta_{\max} = 0$ *then* $\mathrm{MaxLoss}_k(-L) = \infty$ *for all* $k > 0$.

(iii) *Except in cases (i) and (ii),* (2) *has a unique solution* x, *and this* x *equals* $\mathrm{MaxLoss}_k(-L)$.

Existence of MaxLoss

The next theorem solves the cases in which eq. (1) has no positive solution and Theorem 1 does not apply.

Theorem

Eq. (1) *has a unique positive solution* $\overline{\theta}$, *which determines* $\mathrm{MaxLoss}$ *by* $\mathrm{MaxLoss}_k(-L) = \Lambda'(\overline{\theta})$, *except*

- *in case (i) of Theorem 3,*
- *in case (ii) of Theorem 3,*
- *or in case that* $\theta_{\max}, \Lambda(\theta_{\max})$, *and* $\Lambda'(\theta_{\max})$ *are all finite and* $k^2/2 > \theta_{\max}\Lambda'(\theta_{\max}) - \Lambda(\theta_{\max})$. *In this last case*

$$\mathrm{MaxLoss}_k(-L) = (k^2/2 + \Lambda(\theta_{\max}))/\theta_{\max},$$

but there is no generalised scenario achieving $\mathrm{MaxLoss}(k)$.

Outline

1. Traditional Stress Testing
2. Stress Testing with Generalised Scenarios
3. Main Results
4. Applications
5. Connections

Example: one risk factor, normal prior, linear portfolio

Loss linear function of 1 risk factor: $L(r) = l(\mu - r)$,
Risk factor distributed normally with mean μ and variance σ^2.

The worst case scenario $\overline{Q}$ is a normal distribution with variance σ^2 and mean

$$\mu + k\sigma \operatorname{sgn}(l).$$

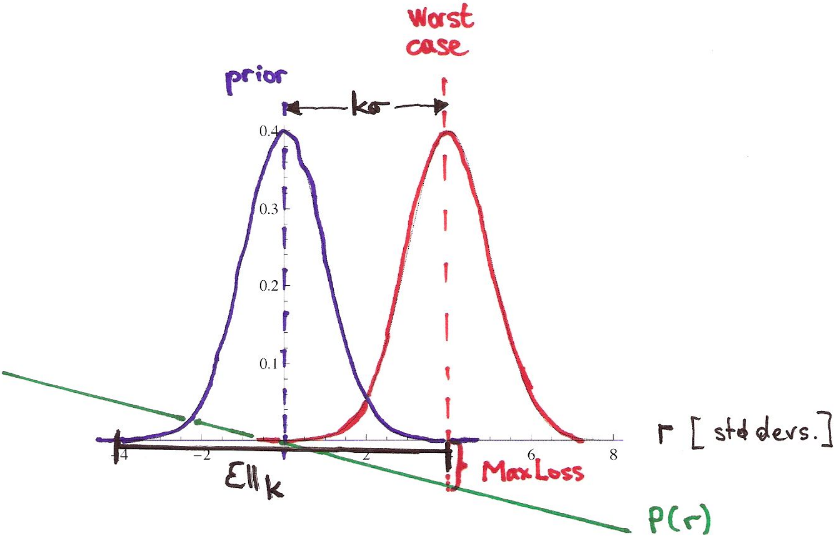
The maximum expected loss is

$$\mathbb{E}_{\overline{Q}}(L) = k\sigma|l|.$$

The Mahalanobis distance between the mean of prior ν and the mean of the generalised worst case scenario equals k:

$$\mathrm{Maha}(\mu, \mu + k\sigma \mathrm{sgn}(l)) = k.$$

Example 1: one risk factor, normal prior, linear portfolio



Example 2: multivariate normal prior, linear portfolio

Loss linear function of n risk factor: $L(\mathbf{r}) = \mathbf{l} \cdot (\boldsymbol{\mu} - \mathbf{r})$,
Risk factor distributed normally: $\mathbf{r} \sim \nu = N(\boldsymbol{\mu}, \Sigma)$.

The worst case scenario is a normal distribution with mean

$$\boldsymbol{\mu} - \frac{k}{\alpha}\Sigma\mathbf{l}$$

and covariance matrix Σ where $\alpha^2 = \mathbf{l}^T\Sigma\mathbf{l}$. The worst case loss is

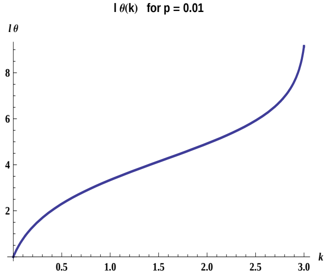
$$\mathbb{E}_{\overline{Q}}(L) = k\alpha.$$

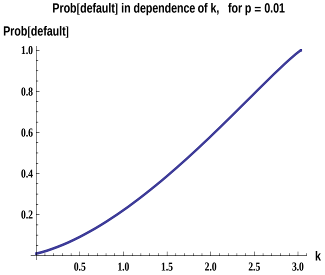
The Mahalanobis distance between the mean of the prior ν and the mean of the generalised worst case scenario equals k:

$$\mathrm{Maha}(\boldsymbol{\mu}, \boldsymbol{\mu} - \frac{k}{\alpha}\Sigma\mathbf{l}) = k.$$

Stressed default probabilities

- $\Omega = \{0,1\} = \{\text{no default}, \ \text{default}\}$.
- $L(0) = 0$ and $L(1) = l$.
- $\nu(\{1\}) = p$: default probability
- $\Lambda(\theta) = \log\left(1 - p + pe^{l\theta}\right)$
- Solve $\frac{\theta p l e^{l\theta}}{1-p+pe^{l\theta}} - \log\left(1 - p + pe^{l\theta}\right) = k^2/2$ numerically for $\overline{\theta}$:





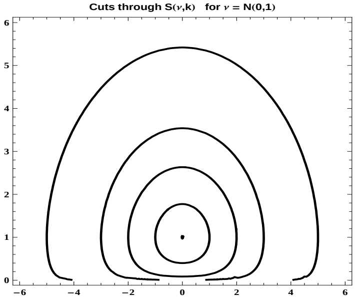
- Worst case default probability: $\frac{pe^{l\overline{\theta}}}{1-p+pe^{l\overline{\theta}}}$.

Model risk

- Model risk stems from the use of inappropriate risk models
- MaxLoss measures model risk related to misspecified risk factor distributions ν:
  $\mathrm{MaxLoss}_k(L) - \mathbb{E}(L)$ gives an upper bound for the amount by which expected losses under alternative risk factor distributions taken from the Kullback-Leibler sphere $S(\nu, k)$ can be worse than expected loss under the prior distribution ν.

Example 1: Volatility model risk

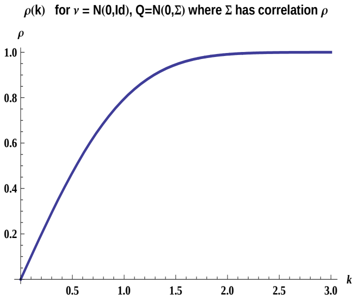
- $S(\nu, k)$ includes alternative risk factor distributions with different volatilities.
- For $\nu = N(0,1)$ here are submanifolds of $S(\nu, k)$ containing normals



- If true μ, σ are in $S(\nu, k)$, the expected loss can be no worse than $\mathrm{MaxLoss}_k(L)$.

Example 2: Correlation model risk

- $S(\nu, k)$ includes alternative risk factor distributions with different correlations.
- Example: $\nu = N(\mathbf{0}, \mathbf{1})$: correlation zero.
- The maximal absolute value of the correlations, for which a normal with mean $\mathbf{0}$ and unit variances is in $S(\nu, k)$:



- If the true correlation is closer to zero than this value, the expected loss can be no worse than $\mathrm{MaxLoss}_k(L)$.

Outline

Convex duality

Duality between risk measure and scenario set:

$$\mathrm{MaxLoss}_k(-L) = \psi^*_{S(\nu,k)}(L),$$

where $\psi_{S(\nu,k)}$ is the indicator function for the set $S(\nu,k) \subset L^1$,

$$\psi_{S(\nu,k)} := \begin{cases} 0 & \text{if } Q \in S(\nu,k) \\ +\infty & \text{otherwise} \end{cases}$$

Fenchel-Legendre duality $\psi^{**}_{S(\nu,k)} = \psi_{S(\nu,k)}$ implies

$$\psi_{S(\nu,k)}(Q) = \sup_{L \in L^\infty(\nu)} \left(\mathbb{E}_Q(L) - \mathrm{MaxLoss}(-L)\right).$$

The Maximum Entropy Principle

- Maximum entropy principle:
  Infer a distribution from a feasible set determined by one or more linear constraints, for example specifying the expectation value of the profit distribution of some fixed portfolio L under the risk factor distribution Q to be inferred.
- In situations where a prior guess ν is available:
  MEP: The distribution to be inferred should be the feasible Q which minimises the relative entropy $I(Q||\nu)$:

$$\min_{Q:\mathbb{E}_Q(L)=m} I(Q||\nu).$$

- Compare this to the problem of determining the generalised worst case scenario: $\sup_{Q:I(Q||\nu)\leq k^2/2} \mathbb{E}_Q(L)$

Asset Pricing

- Asset pricing in arbitrage-free incomplete markets:
  From a prior, the real world distribution ν, and observed prices of primary assets one determines a risk neutral distribution Q.
- Risk-neutrality of measure Q: $\mathbb{E}_Q(\mathbf{r}) = \mathbf{r}_{\mathrm{CM}}$.
  Observed prices of primary assets are reproduced as expectations with respect to Q of discounted future cash-flows.
- Then use Q to price derivatives of primary assets by $\mathbb{E}_Q(-L)$.
- MEP: Choose risk-neutral Q with minimal rel. entropy with respect to real world probability probability ν. This is

$$\frac{dQ_{\mathbf{l}^*}}{d\nu}(\mathbf{r}) := e^{\mathbf{l}^* \cdot \mathbf{r} - \Lambda(1, \mathbf{l}^*)},$$

  where $\mathbf{l}^*$ satisfies $\nabla_{\mathbf{l}} \Lambda(1, \mathbf{l}^*) = \mathbf{r}_{\mathrm{CM}}$.
- Compare this to the generalised worst case scenario $\frac{d\overline{Q}}{d\nu}(\mathbf{r}) = e^{\overline{\theta} L(\mathbf{r}) - \Lambda(\overline{\theta}, L)}$ where $\partial \Lambda(\overline{\theta}, L)/\partial \theta = \mathrm{MaxLoss}_k(-L)$.

Portfolio Optimisation

- Among the portfolios which can be purchased with an initial budget w identify the one with highest expected utility:

$$\max_{L:\mathbb{E}_{\nu^*}(-L)\leq w} \mathbb{E}_{\nu}(u(-L)).$$

- Assumes some risk neutral pricing measure ν^*.
- The optimal portfolio L^* satisfies a first order condition:

$$\mathbb{E}_{\nu}(u'(-L^*)(-L-\mathbb{E}_{\nu^*}(-L))=0 \qquad (3)$$

for all bounded portfolios $L \in L^{\infty}(\nu)$.

- The optimal portfolio L^* determines a risk-neutral Q^* by

$$\frac{dQ^*}{d\nu}(\mathbf{r}) := \frac{u'(-L^*(\mathbf{r}))}{\mathbb{E}_{\nu}(u'(-L^*))}.$$

Portfolio Optimisation

- Assume exponential utility $u(x) = 1 - \exp(-\alpha x)$.
- Then optimal portfolio $L^*(\mathbf{r}) = \frac{1}{\alpha}\frac{d\nu^*}{d\nu}(\mathbf{r}) - w - \frac{1}{\alpha}I(\nu^*||\nu)$.
- The risk neutral Q^* derived from the optimal portfolio L^* is

$$\frac{dQ^*}{d\nu}(\mathbf{r}) = e^{\alpha L^*(\mathbf{r}) - \Lambda(\alpha, L^*)}.$$

Compare this to the generalised worst case scenario $\frac{d\overline{Q}}{d\nu}(\mathbf{r}) = e^{\overline{\theta}L(\mathbf{r}) - \Lambda(\overline{\theta}, L)}$.

Proposition

The risk neutral measure Q^* determined from the optimal portfolio L^* for exponential utility coincides with the pricing measure: $Q^* = \nu^*$.

Microeconomic Equilibrium

- Do not assume pricing measure ν^* but derive it from microeconomic equilibrium.
- Equilibrium: Prices and demands of agents adapt to each other so that each agent solves his utility maximisation problem and markets clear.
- Assume: agents $a \in A$ have exponential utility and initial endowment $W_a \in L^\infty(\Omega, \mathbb{F}, \nu)$.
- Then the prices in equilibrium are given by:

$$\frac{d\nu^*}{d\nu}(\mathbf{r}) = e^{\alpha W(\mathbf{r}) - \Lambda(\alpha, W)},$$

  where $\alpha = 1/\sum_{a\in A}(1/\alpha_a)$ and
  W is the aggregated supply of assets (the market portfolio).
- In equilibrium the portfolio optimal for agent $a \in A$ is a linear share in the market portfolio, in inverse proportion to her risk aversion.