Providing for the Worst: Generalised Worst Case Scenarios and Maximum Loss

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Outline

1. Maximum Loss
2. Overcoming dimensional dependence
3. Overcoming coordinate dependence (with I. Csiszár)

Traditional Maximum Loss

$$\mathrm{MaxLoss}_A(P) := (ess) \sup_{\mathbf{r} \in A} (EP - P(\mathbf{r}))$$

where P is the portfolio value function
EP is the expected value of the portfolio
A is some admissibility domain, consisting of point scenarios

Why use MaxLoss?

- MaxLoss is a coherent risk measure,
  in particular it is subadditive:

  $$\mathrm{MaxLoss}_A(P_1 + P_2) \leq \mathrm{MaxLoss}_A(P_1) + \mathrm{MaxLoss}_A(P_2)$$

  for all portfolios P_1, P_2.
- MaxLoss is a systematic way of stress testing
- MaxLoss specifies the dangerous scenarios
- Worst case scenario indicates targeted counteraction

MaxLoss is systematic stress testing

- A: admitted stress scenarios
  "scenarios above some plausibility threshold"
- No executive dilemma:
  "What to do about alarming stress test results in implausible scenarios?"
- No false sense of safety:
  "Did our stress tests miss dangerous but plausible scenarios?"

Coherent risk measures are MaxLoss

Theorem (Delbaen)

Every coherent risk measure ρ is of the form

$$\rho(P) = EP - \min_{Q \in A} \mathbb{E}_Q(P)$$

for some admissibility domain A
of generalised scenarios (probability distributions) Q .

Idea: duality

Outline

The problem of dimensional dependence

Assume normal distribution of **r**.
Traditionally, one takes

$$A := \mathrm{Ell}_\alpha \quad := \quad \{\mathbf{r} : \sqrt{(\mathbf{r}-\boldsymbol{\mu})^T \cdot \Sigma^{-1} \cdot (\mathbf{r}-\boldsymbol{\mu})} \leq k_\alpha\}$$
$$=: \quad \{\mathbf{r} : \mathrm{Maha}(\mathbf{r}) \leq k_\alpha\}$$

where k_α is chosen to give

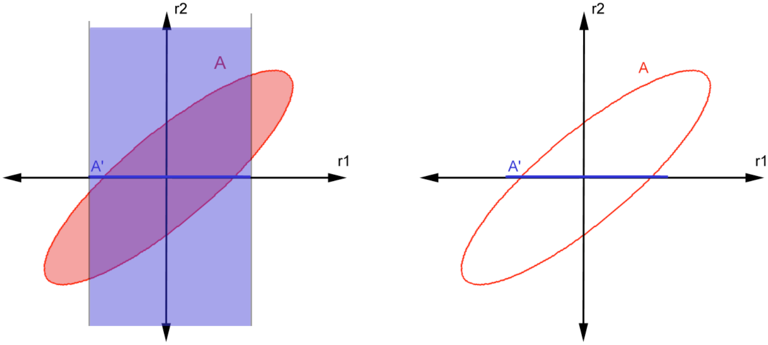
$$Prob(\mathrm{Ell}_\alpha) = \alpha$$

from

$$Prob(\mathrm{Ell}_\alpha) = \frac{2^{-n/2}}{\Gamma(\frac{n}{2})} \int_0^{k_\alpha^2} t^{n/2-1} \exp(-\frac{t}{2}) dt = F_{\chi^2_n}(k_\alpha^2). \qquad (1)$$

k_α, and thus MaxLoss, depends on the number of risk factors, which is to some degree arbitrary (e.g. empty risk factors).

Dimensional dependence of MaxLoss



The two-dimensional admissibility domain A and the one-dimensional A' have the same probability, but different MaxLoss.

Dimensional dependence of MaxLoss

	$k_\alpha \sim \mathrm{MaxLoss}$ if P linear			
α	n=1	n=5	n=10	n=20
0.95	1.9630	2.9344	3.7442	4.9493
0.99	2.0457	2.9587	3.7597	4.9595
0.999	2.0643	2.9641	3.7631	4.9617

Risk modellers are free to include or exclude
empty risk factors (on which the portfolio does not depend at the moment) or risk factors which are highly correlated to other risk factors already included in the model.

The problem: Inclusion of empty or highly correlated risk factors increases MaxLoss if $A = \mathrm{Ell}_\alpha$.

Overcoming dimensional dependence of MaxLoss

Basic idea:
Instead of $A := \mathrm{Ell}_\alpha$ take

$$A := \mathrm{Ell}_k := \{\mathbf{r} : \mathrm{Maha}(\mathbf{r}) \leq k\}$$

k: Mahalanobis radius of the ellipsoid Ell_k.

Intuitively: k is maximal size of moves,
measured in "multivariate standard deviations".

Empty risk factors

- Assume P is a function of n risk factors which does not depend on the coordinate r_n.
- For $\mathbf{r} = (r_1, \ldots, r_n)$ define $\mathbf{r}' := (r_1, \ldots, r_{n-1})$.
- Define the function P' on $\mathbb{R}^{n-1}$ by $P'(\mathbf{r}') := P(r_1, \ldots r_{n-1}, r_n)$ where r_n takes an arbitrary value.
- Define the (n-1)-dimensional ellipsoid $\mathrm{Ell}'_k := \{\mathbf{r}' : \mathrm{Maha}(\mathbf{r}') \leq k\}$.

Theorem

Then we have

$$\operatorname*{ess\,inf}_{\mathbf{r}' \in \mathrm{Ell}'_k} P'(\mathbf{r}') = \operatorname*{ess\,inf}_{\mathbf{r} \in \mathrm{Ell}_k} P(\mathbf{r}). \tag{2}$$

This implies $\mathrm{MaxLoss}_{\mathrm{Ell}_k}(P) = \mathrm{MaxLoss}_{\mathrm{Ell}'_k}(P')$.

Highly correlated risk factors

Theorem

- *Assume the portfolio value P is a continuous function of n variables.*
- *Denote by P' the function*

$$P'(\mathbf{r}') := P(\mathbf{r}', r_n^*),$$

where r_n^* is the conditional expectation of risk factor r_n given $\mathbf{r}'$.

- *Assume that risk factor r_n is strongly positively or negatively correlated with some other risk factor r_i. Denote by $\rho(i,n)$ the correlation coefficient between r_n and r_i.*

Then

$$\lim_{\rho(i,n)\to\pm 1} \mathrm{MaxLoss}_{\mathrm{Ell}_k}(P) = \mathrm{MaxLoss}_{\mathrm{Ell}'_k}(P'). \qquad (3)$$

The effect on MaxLoss of adding or removing a risk factor goes to zero if the correlation of this risk factor to some existing risk factor goes to ± 1.

Outline

Coordinate dependence of MaxLoss

There are transformations mapping the normal into the normal distribution, but not mapping the ellipsoid into the ellipsoid. Dilemma:

- Either: single out one specific coordinate system and take as admissibility domain in other coordiate systems the transform of the ellipsoid, even if the transform is itself not an ellipsoid.
  It is weird to take a non-elliptical admissibilty domain in transformed coordinates, in which risk factors are distributed normally.
- Or: we can take as admissibility domain in the transformed coordinates the ellipsoids. But this will result in a different worst case scenario and in a different MaxLoss value than in the original coordinates.

Generalised MaxLoss

- Instead of point scenarios consider generalised scenarios (distributions).
- Instead of Mahalanobis radius take I-divergence from prior distribution ν:

$$I(Q||\nu) := \begin{cases} \int \log \frac{dQ}{d\nu}(\mathbf{r})dQ(\mathbf{r}) & \text{if } Q \ll \nu \\ +\infty & \text{if } Q \not\ll \nu \end{cases}$$

- Instead of ellipsoid take Kullback-Leibler sphere in the space of distributions $S(\nu, k) := \{Q : I(Q||\nu) \leq k^2/2\}$.
- Instead of $P(\mathbf{r})$ take $\mathbb{E}_Q(P)$ or $\mathbb{E}_Q(L)$ with $L := EP - P$.

$$\mathrm{MaxLoss}_k P := EP - \min_{Q \in S(\nu,k)} \mathbb{E}_Q(P) = \max_{Q \in S(\nu,k)} \mathbb{E}_Q(L).$$

$A = S(\nu, k)$ in Delbaen's representation of coherent risk measures.

Calculation of generalised MaxLoss

Theorem

The generalised worst case scenario $\overline{Q}$ is the distribution with density

$$\frac{d\overline{Q}}{d\nu}(\mathbf{r}) := e^{\theta L(\mathbf{r}) - \Lambda(\theta)}$$

where θ is the positive solution of

$$\theta \Lambda'(\theta) - \Lambda(\theta) = k^2/2 \tag{4}$$

and

$$\Lambda(\theta) := \log\left(\int \exp(\theta L(\mathbf{r})) d\nu(\mathbf{r}) \right).$$

The Maximum Loss achieved in the generalised worst case scenario $\overline{Q}$ is

$$\mathbb{E}_{\overline{Q}}(L) = \Lambda'(\theta).$$

Existence of solution for generalised MaxLoss

Theorem

For k small enough th equation

$$\theta\Lambda'(\theta) - \Lambda(\theta) = k^2/2$$

has a unique positive solution.

Example: one risk factor, normal prior, linear portfolio

Loss linear function of one risk factor: $L(r) = l(\mu - r)$,
Risk factor distributed normally: $r \sim \nu = N(\mu, \sigma^2)$.

The worst case scenario $\overline{Q}$ is a normal distribution with variance σ^2 and mean

$$\mu - k\sigma.$$

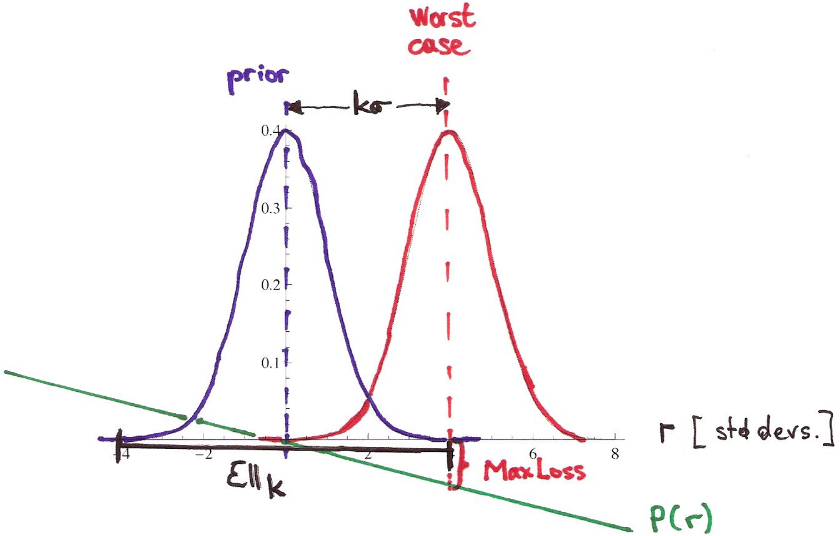
The maximum expected loss is

$$\mathbb{E}_{\overline{Q}}(L) = k\sigma l.$$

The Mahalanobis distance between the mean of the prior distribtion and the mean of the generalised worst case scenario equals k:

$$\mathrm{Maha}(\mu, \mu - k\sigma) = k.$$

Example 1: one risk factor, normal prior, linear portfolio



Example 2: multivariate normal prior, linear portfolio

Loss linear function of n risk factor: $L(\mathbf{r}) = \mathbf{l} \cdot (\boldsymbol{\mu} - \mathbf{r})$,
Risk factor distributed normally: $\mathbf{r} \sim \nu = N(\boldsymbol{\mu}, \Sigma)$.

The worst case scenario is a normal distribution with mean

$$\boldsymbol{\mu} - \frac{k}{\alpha}\Sigma\mathbf{l}$$

(where $\alpha^2 = \mathbf{l}^T\Sigma\mathbf{l}$) and covariance matrix Σ . The worst case loss is

$$\mathbb{E}_{\overline{Q}}(L) = k\alpha.$$

The Mahalanobis distance between the mean of the prior distribtion and the mean of the generalised worst case scenario equals k:

$$\mathrm{Maha}(\boldsymbol{\mu}, \boldsymbol{\mu} - \frac{k}{\alpha}\Sigma\mathbf{l}) = k.$$

Generalised MaxLoss reduces to traditional MaxLoss:

$$\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) = \max_{\mathbf{r} \in \mathrm{Ell}_k} L(\mathbf{r}).$$

Application

- Coordinate-free definition of MaxLoss
- Definition of MaxLoss for non-normal priors
- Definition of MaxLoss for discrete risk factor distributions along the same lines