A Theory of General Stress Testing

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Outline

Purpose of Stress Testing: Complement statistical risk measurement

- Statistical risk measurements: What are probs of big losses?
  Stress Testing: Which scenarios lead to big losses?
  Derive risk reducing action.
- Statistical risk measurement: Assume fixed model.
  Stress Testing: Consider alternative risk factor distribution.
  Address model risk.

Requirements on stress scenarios (Basel II)

- Plausibility
- Severity
- Suggestive of risk reducing action

Stress Testing with Point Scenarios

Framework:
Prior risk factor distribution ν on $(\Omega, \mathbb{F})$.
Portfolio loss function L on Ω, measurable.

Stress Testing:
Find worst case scenario and worst case loss
in some set $A \subset \Omega$ of point scenarios:

$$\sup_{\mathbf{r} \in A} L(\mathbf{r}) =: \rho_A(-L).$$

Risk measurement:
$\rho_A(-L)$ is a coherent risk measure.

Systematic Stress Testing with Point Scenarios

- For elliptical risk factor distribution ν, introduce **measure of plausibility** for point scenarios:

$$\mathrm{Maha}(\mathbf{r}) := \sqrt{(\mathbf{r} - \mathbb{E}(\mathbf{r}))^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - \mathbb{E}(\mathbf{r}))},$$

  where Σ is covariance matrix of risk factor distribution ν.
- Choose as scenario set A

$$\mathrm{Ell}_k := \{\mathbf{r} : \mathrm{Maha}(\mathbf{r}) \leq k\},$$

  where k is either determined so as to give the ellipsoid some desired probability mass or independent of the number of risk factors.

Advantages of Systematic Stress Testing with Point Scenarios

All three requirements on stress testing are met:

- Do not miss plausible but severe scenarios.
- Do not consider scenarios which are too implausible.
- Worst case scenario over Ell_k gives information about portfolio structure and suggests risk reducing action.

Problems of Systematic Stress Testing with Point Scenarios

1. Maha does not take into account fatness of tails.
2. How choose scenario set for non-elliptical risk factor distributions ν?
3. $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$ depends on choice of coordinates.
4. $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$ is not law-invariant: Portfolios L_1, L_2 might have the same profit/loss distribution but different $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$.
5. Model risk is not addressed.

Outline

Generalised Scenarios

- Generalised scenario: Probability distribution on $(\Omega, \mathbb{F})$.
- Interpretation 1:
  Point scenarios are generalised scenarios with support concentrated on one point.
- Interpretation 2:
  Risk factor distributions alternative to the prior ν.
  Model risk.

Systematic stress testing with generalised scenarios

- Measure of plausibility: relative entropy
  (I-divergence, information gain, Kullback-Leibler distance)

$$I(Q||\nu) := \begin{cases} \int \log \frac{dQ}{d\nu}(\mathbf{r}) dQ(\mathbf{r}) & \text{if } Q \ll \nu \\ +\infty & \text{if } Q \not\ll \nu \end{cases}$$

- Scenario set: Instead of ellipsoid take
  Kullback-Leibler sphere in the space of distributions

$$S(\nu, k) := \{Q : I(Q||\nu) \leq k^2/2\}.$$

- Severity of scenarios: Instead of $L(\mathbf{r})$ take $\mathbb{E}_Q(L)$
- Generalised MaxLoss:

$$\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) =: \mathrm{MaxLoss}_k(L)$$

Advantages of Systematic Stress Testing with Generalised Scenarios

1. Relative entropy does take into account fatness of tails of ν.
2. Scenario set is naturally defined for non-elliptical risk factor distributions ν.
3. $\mathrm{MaxLoss}_k$ does not depend on choice of coordinates.
4. $\mathrm{MaxLoss}_k$ is law-invariant: Portfolios L_1, L_2 with the same profit/loss distribution have the same $\mathrm{MaxLoss}_k$.
5. Model risk is addressed:
   Generalised scenarios are alternatives to prior risk factor distribution ν.

Outline

Calculation of generalised MaxLoss

Tool from large deviations theory for solving explicitly the optimisation problem $\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L)$:

$$\Lambda(\theta, L) := \log \left(\int e^{\theta L(\mathbf{r})} d\nu(\mathbf{r}) \right).$$

$\theta_{\max} := \sup\{\theta : \Lambda(\theta) < +\infty\}$

Calculation of generalised MaxLoss

Theorem

The generalised worst case scenario $\overline{Q}$ is the distribution with ν-density

$$\frac{d\overline{Q}}{d\nu}(\mathbf{r}) := \frac{e^{\overline{\theta}L(\mathbf{r})}}{\int e^{\overline{\theta}L(\mathbf{r})}\,d\nu(\mathbf{r})} = e^{\overline{\theta}L(\mathbf{r})-\Lambda(\overline{\theta})},$$

where $\overline{\theta}$ is the positive solution of

$$\theta\Lambda'(\theta) - \Lambda(\theta) = k^2/2, \qquad (1)$$

provided the solution exists. The generalised Maximum Loss achieved in the generalised worst case scenario $\overline{Q}$ is

$$\mathrm{MaxLoss}_k(-L) = \mathbb{E}_{\overline{Q}}(L) = \Lambda'(\overline{\theta}).$$

MaxLoss from convex conjugate

If Theorem 1 applies, then

$$\Lambda^*(x) = k^2/2 \text{ and } x > \mathbb{E}_\nu(L), \tag{2}$$

where $\Lambda^*(x)$ is the convex conjugate of $\Lambda(\theta)$.

Theorem

(i) *If* $\operatorname{ess\,sup}(L)$ *is finite, and* $k^2/2 \geq -\log(\nu(\{\mathbf{r} : L(\mathbf{r}) = \operatorname{ess\,sup}(L)\}))$, *then* $\mathrm{MaxLoss}_k(-L) = \operatorname{ess\,sup}(L)$. *Neither eq.* (2) *nor eq.* (1) *has a solution.*

(ii) *If* $\theta_{\max} = 0$ *then* $\mathrm{MaxLoss}_k(-L) = \infty$ *for all* $k > 0$. *Neither eq.* (2) *nor eq.* (1) *has a solution.*

(iii) *Except in cases (i) and (ii),* (2) *has a unique solution* x, *and this* x *equals* $\mathrm{MaxLoss}_k(-L)$.

Existence of MaxLoss

The next theorem solves the cases in which eq. (1) has no positive solution and Theorem 1 does not apply.

Theorem

Eq. (1) *has a unique positive solution* $\overline{\theta}$, *which determines* $\mathrm{MaxLoss}$ *by* $\mathrm{MaxLoss}_k(-L) = \Lambda'(\overline{\theta})$, *except*

- *in case (i) of Theorem 2,*
- *in case (ii) of Theorem 2,*
- *or in case that* $\theta_{\max}, \Lambda(\theta_{\max})$, *and* $\Lambda'(\theta_{\max})$ *are all finite and* $k^2/2 > \theta_{\max}\Lambda'(\theta_{\max}) - \Lambda(\theta_{\max})$. *In this last case*

$$\mathrm{MaxLoss}_k(-L) = (k^2/2 + \Lambda(\theta_{\max}))/\theta_{\max},$$

but there is no generalised scenario achieving $\mathrm{MaxLoss}(k)$.

Outline

Example: one risk factor, normal prior, linear portfolio

Loss linear function of 1 risk factor: $L(r) = l(\mu - r)$,
Risk factor distributed normally with mean μ and variance σ^2.

The worst case scenario $\overline{Q}$ is a normal distribution with variance σ^2 and mean

$$\mu + k\sigma \operatorname{sgn}(l).$$

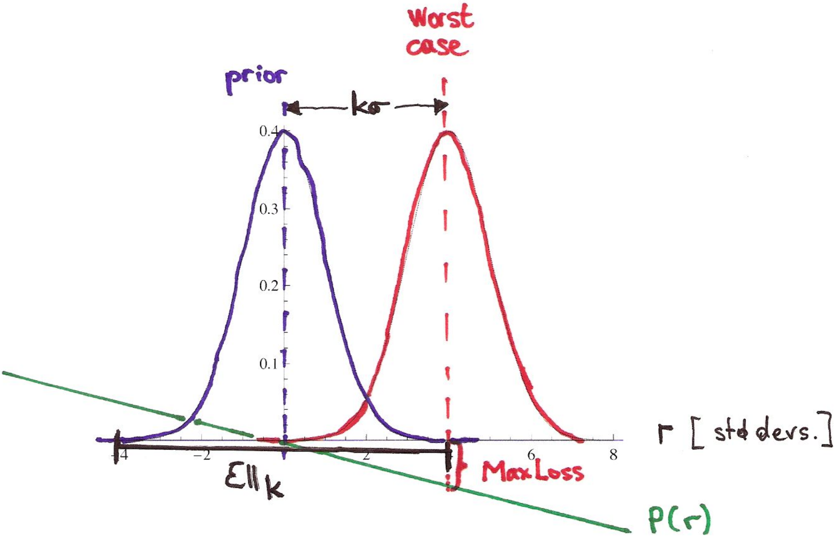
The maximum expected loss is

$$\mathbb{E}_{\overline{Q}}(L) = k\sigma|l|.$$

The Mahalanobis distance between the mean of prior ν and the mean of the generalised worst case scenario equals k:

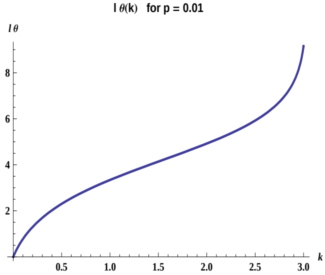
$$\mathrm{Maha}(\mu, \mu + k\sigma \mathrm{sgn}(l)) = k.$$

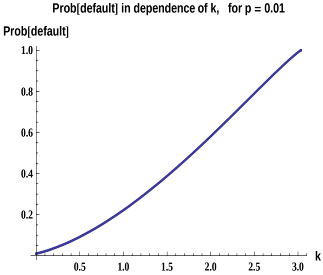
Example 1: one risk factor, normal prior, linear portfolio



Stressed default probabilities

- $\Omega = \{0, 1\} = \{\text{no default}, \text{ default}\}$.
- $L(0) = 0$ and $L(1) = l$.
- $\nu(\{1\}) = p$: default probability
- $\Lambda(\theta) = \log\left(1 - p + pe^{l\theta}\right)$
- Solve $\frac{\theta p l e^{l\theta}}{1-p+pe^{l\theta}} - \log\left(1 - p + pe^{l\theta}\right) = k^2/2$ numerically for $\overline{\theta}$:





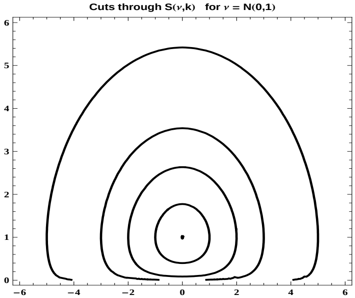
- Worst case default probability: $\frac{pe^{l\overline{\theta}}}{1-p+pe^{l\overline{\theta}}}$.

Model risk

- Model risk stems from the use of inappropriate risk models
- MaxLoss measures model risk related to misspecified risk factor distributions ν:
  $\mathrm{MaxLoss}_k(L) - \mathbb{E}(L)$ gives an upper bound for the amount by which expected losses under alternative risk factor distributions taken from the Kullback-Leibler sphere $S(\nu, k)$ can be worse than expected loss under the prior distribution ν.

Example 1: Volatility model risk

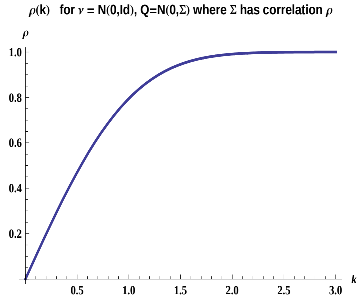
- $S(\nu, k)$ includes alternative risk factor distributions with different volatilities.
- For $\nu = N(0,1)$ here are submanifolds of $S(\nu, k)$ containing normals



- If true μ, σ are in $S(\nu, k)$, the expected loss can be no worse than $\mathrm{MaxLoss}_k(L)$.

Example 2: Correlation model risk

- $S(\nu, k)$ includes alternative risk factor distributions with different correlations.
- Example: $\nu = N(\mathbf{0}, \mathbf{1})$: correlation zero.
- The maximal absolute value of the correlations, for which a normal with mean $\mathbf{0}$ and unit variances is in $S(\nu, k)$:



- If the true correlation is closer to zero than this value, the expected loss can be no worse than $\mathrm{MaxLoss}_k(L)$.