Portfolio Selection with Transaction Costs under Coherent Dynamic Risk Constraints

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Outline

Motivation

- Maximise long-term expected return
- of a portfolio of riskless bond and m stocks
- under transaction costs
- and coherent multi-period risk constraints

Dynamics of Bond and Stock Prices

$$
\begin{aligned}
dS_0(t) &= S_0(t) r dt \\
dS_i(t) &= S_i(t) \left(\mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dB_j(t) \right), \qquad i = 1, \dots, m
\end{aligned}
$$

B: *m*-dimensional Brownian motion
with covariance matrix $\sigma' \sigma$.
μ_i average return of stock *i*
r interest rate of bond, assumed constant

Portfolio Selection without Transaction Costs (Morton Pliska 1995)

- $(V_0, V_1, \ldots V_m)$: value of bond and stock holdings in portfolio depending on trading strategy π and initial endowment $(x_0, x_1, \ldots, x_m)$
- Maximise expected long-term return

$$\lim_{t\to\infty} \frac{E(\log((V^{\pi,x}(t)))}{t}$$

by choosing fractions $\pi_i(t) := V_i(t)/V_0(t)$ such that

$$\frac{\pi^*}{1+\pi^*\cdot 1} = (\sigma\sigma')^{-1}(\mu - r1).$$

Portfolio Selection without Transaction Costs (Morton Pliska 1995)

As in Merton (1969) optimisation of expected utility of consumption:

- Fractions of wealth in various stocks time-independent.
- This requires constant trading.

Proportional Transaction Costs

- Buying one euro worth of stock i will cost $(1 + \lambda_{bi})$ euro in cash from the bond.
- Selling one euro worth of stock i will result in $(1 - \lambda_{si})$ euro in cash that is added to the bond.
- $\lambda_{si}, \lambda_{si} \geq 0$

Controlled Value Dynamics under Proportional Transaction Costs

$$t^{-1}E(\ln w(t)) = t^{-1}\ln w(0) + t^{-1}E\Big[\int_0^t \Big(\frac{r}{1+\pi(s)\cdot 1} + \frac{\mu\cdot\pi(s)}{1+\pi(s)\cdot 1} - \frac{\pi(s)\sigma\sigma'\pi(s)'}{2(1+\pi(s)\cdot 1)^2}\Big)ds - \int_0^t \frac{\lambda_s * \pi(s')}{1+\pi(s')\cdot 1}\cdot dL(s') - \int_0^t \frac{\lambda_b}{1+\pi(s)\cdot 1}\cdot dR(s)\Big].$$

Control processes

- $(L_0(t), L_1(t), \ldots, L_m(t))$ selling processes
- $(R_0(t), R_1(t), \ldots, R_m(t))$ buying processes

together with price dynamics determine the total portfolio value process $w(t)$.

Limit Control Strategies

In the presence of transaction costs continuous trading is not optimal.

Control limits $[A_i, B_i]$ for each stock i:

- If π_i is below A_i buy the minimal amount of stock necessary to bring π_i back to A_i.
- If π_i is above B_i sell the minimal amount of stock necessary to bring π_i back to B_i.
- If $\pi_i \in [A_i, B_i]$ do nothing with stock i.

No transaction region: all $\pi_i \in [A_i, B_i]$.

One-Period Risk Measures

Intuition:
Risk capital of a portfolio = capital necessary to safely hold portfolio.

- V: random variable on prob space $(\Omega, \mathcal{F}, P)$ describing portfolio values at end of time horizon.
- Risk measure $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$
- $\rho(V)$ is capital necessary to safely hold portfolio V

Convex and Coherent One-Period Risk Measures

- A risk measure ρ is convex if
  - if $V \geq 0$ then $\rho(V) \leq 0$.
  - $\rho(V + a) = \rho(V) - a$ for every constant (riskfree) function a
  - $\rho(V + W) \leq \rho(V) + \rho(W)$: subadditivity (diversification)
  - Fatou property
- A risk measure is coherent if additionally
  - $\rho(\lambda V) = \lambda\rho(V)$: no liquidity risk

Convex and Coherent One-Period Risk Measures

- Examples of coherent risk measures:
  Exptected Shortfall, Maximum Loss, Greeks,
  semi-variance
- Value at Risk is not coherent
  if risk factors are not elliptically distributed
  or if portfolio is not a linear function of the risk factors.

Multi-Period Risk Measures

Risk is measured not just at one time horizon,
but at regular intervals
or even continuously.

Risk is defined for value processes $V : [0, \infty] \times \Omega \to \mathbb{R}$ which

- are $\mathcal{F}_t$ adapted
- left continuous with right limits
- $|| \sup_{0 \le t \le T} |V_t| \, ||_\infty < \infty$ for all T

Multi-Period Risk Measures

A coherent one-period risk measure ρ_0
gives rise to a coherent multi-period risk measure ρ by

$$\rho(X) = \rho_0\left(\inf_{t\in[0,\infty]} X_t\right).$$

Risk Constraint for Controlled Portfolio

Admissibility of the strategy π starting from x:

$$\rho_0(V^{\pi,x}(t)) \leq 0$$

for all times t.
Then risk capital is always sufficient.

Cone Formulation of Risk Constraint

- To the risk measure ρ_0 there corresponds a cone $C \subset L^\infty(\Omega, \mathcal{F}, P)$:

$$C = \{V : \rho(V) \leq 0\}.$$

- A strategy $\pi_t \leftrightarrow (L_t, R_t)$ is acceptable for a starting point $x \in C$
if and only if $V^{\pi,x}$ remains in C for all times.

Cone Formulation of Risk Constraint

- The net value of the portfolio, after transfer of all stock wealth to the bond is

$$W(\overline{V}) := V_0 + \sum_{i=1}^{m} \min\Big[(1-\lambda_{si})V_i, (1+\lambda_{bi})V_i\Big]$$

- Define $K \subset \mathbb{R}^{m+1}$ by

$$K := \{r \in \mathbb{R}^{m+1} | W(r) \in C\}.$$

  K contains the portfolios $V(t)$ for which the net value after transfer of stock wealth to the bond still has negative risk.

- $\mathcal{A}_x^\rho$: the set of all strategies for which $\rho_0(V^{\pi,x}(t)) \leq 0$ for the starting point x and all t.

Stochastic Control Problem

Find control processes $L(t), R(t)$ which

- maximises the objective function

$$\liminf_{T\to\infty} t^{-1} E\Big[\int_0^t h(\pi(s))ds - \int_0^t g_s(\pi(s)) \cdot dL(s) - \int_0^t g_b(\pi(s)) \cdot dR(s)\Big]$$

- under the condition that the controlled process $\pi(t)$ follows

$$\begin{aligned}\pi(t) = \pi(0) + \int_0^t \pi * (\mu - r1)ds + \int_0^t \pi * (\sigma dB(s)) \\ + \int_0^t \pi(s) * ((1-\lambda_b) \cdot dR(s)) + \int_0^t dR(s) \\ + \int_0^t \pi(s)^2 * ((1-\lambda_s) \cdot dL(s)) + \int_0^t \pi(s) * dL(s).\end{aligned}$$

- and the net value process

$$W(\pi(t)) := V_0(t)\Big(1 + \sum_{i=1}^{m} \min\left[(1-\lambda_{si})\pi_i(t), (1+\lambda_{bi})\pi_i(t)\right]\Big)$$

The HJB Equation

For optimal control strategy with value d, the function $J(x) := \frac{E(\ln w(t))}{t} - td$ satisfies

$$\max\Big[\Gamma J(x) - h(x) + d, \max_{1\leq i\leq m} -\frac{\partial J}{\partial x_i} + \frac{\lambda_{si}}{(1 + x\cdot 1)(x_i(1-\lambda_{si})+1)}, \max_{1\leq i\leq m} \frac{\partial J}{\partial x_i} + \frac{\lambda_{bi}}{(1 + x\cdot 1)(x_i(1-\lambda_{bi})+1)}\Big] = 0$$

for $x \in K$. For x on the boundary ∂K, $J(x) = 0$.

$$\begin{aligned} \Gamma &:= \frac{1}{2}(x\nabla)\sigma\sigma'(x\nabla)' + (r,\mu)\cdot\nabla + d \\ h(x) &:= \frac{r}{1+x\cdot 1} + \frac{\mu\cdot x}{1+x\cdot 1} - \frac{x\sigma\sigma' x'}{2(1+x\cdot 1)^2} \end{aligned}$$

Results

Theorem

Assume there exists a classical or viscosity solution $(J(x), d)$ of the HJB equation, and there exist intervals $[A_i, B_i]$ such that

$$\Gamma J(x) - h((x) + d = 0 \qquad \text{if all } x_i \in [A_i, B_i]$$

$$-\frac{\partial J}{\partial x_i} + \frac{\lambda_{si}}{(1 + x \cdot 1)(x_i(1 - \lambda_{si}) + 1)} = 0 \qquad \text{if } x_i \geq B_i$$

$$\frac{\partial J}{\partial x_i} + \frac{\lambda_{bi}}{(1 + x \cdot 1)(x_i(1 - \lambda_{bi}) + 1)} = 0 \qquad \text{if } x_i \leq A_i.$$

Then there exists an optimal strategy solving the stochastic control problem, and this strategy is the control limit strategy with control limits $[A_i, B_i]$.

Results

Theorem

Under some technical conditons the HJB-equation has a unique viscosity solution.

Along the lines of Kabaov and Klüppelberg (2004).

Outlook

Next steps:

Development and evaluation of algorithms

- solving the HJB equation
- solving the stochastic control problem