A Gödel-Turing Perspective on Quantum States Indistinguishable from Inside*

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Abstract

By a diagonalisation argument, Bell states are not distinguishable from inside. This result is closely related to the theorems of Gödel, Church, and Turing in spite of important dissimilarities.

1 Introduction

Consider two entangled quantum systems A, B. Without loss of generality take A and B to be qubits. A and B are allowed to communicate on a classical channel the results of measurements performed by each of them. Can A or B, by themselves or in cooperation, immediately or with hindsight, discriminate between two different entangled states of the joint system $A\&B$? We argue that they cannot discriminate states ρ_1, ρ_2 of $A\&B$ whose partial trace over both A and B coincide, i.e. for which $\mathrm{tr}_A(\rho_1) = \mathrm{tr}_A(\rho_2)$ and $\mathrm{tr}_B(\rho_1) = \mathrm{tr}_B(\rho_2)$. These conditions are satisfied for example for the density matrices of the Bell states

$$\rho_1 = (|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|)/2 \quad (1)$$

$$\rho_2 = (|00\rangle\langle 11| - |11\rangle\langle 00| - |00\rangle\langle 11| + |11\rangle\langle 11|)/2 \quad (2)$$

(There are two more Bell states, which we will not use here. Our indistinguishability results are valid for any two of the four Bell states, and in fact

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for any two quantum states differing only in the entanglement between A and B.)

These internal indistinguishability results differ from other well-known restrictions of the distinguishability of quantum states. One such argument establishes that there is no quantum procedure to reliably distinguish non-orthogonal states, see e.g. Nielsen and Chuang [1, p.87]. Holevo [2] established an upper bound on the information accessible in quantum systems.

Our result is an application of a more general argument [3] establishing limitations on measurements from inside. In order to keep the treatment self-contained, we give a sketch of that argument in Section 2. That argument uses a diagonalisation procedure. Measurements from inside are self-referential because they provide information about a larger system, which in turn has implications for the observer (or the apparatus) contained in it. Certain final states of the large observed system are impossible because they imply paradoxical self-reference.

It is a misconception that the resulting limitations on measurements from inside are due to the apparatus being "smaller" than the observed system containing it, and that the small apparatus could not discriminate all states of the observed system. Size is not the reason for the limitations on measurements from inside. A smaller system could discriminate all states of an external system with more degrees of freedom as long as the state spaces have the same cardinality. If the observed system is external in the sense that it does not contain the apparatus, a measurement does not give rise to self-reference and nothing prevents a discrimination of all states.

For *quantum* systems the limitations on measurements from inside are more serious than for classical systems. If system A or B is classical, then A could measure the state of B without limitations, and B could measure the state of A. By communicating their measurement results they could determine uniquely the joint state of $A\&B$. In the *quantum* scenario, however, entanglement prevents the unique determination of the state of $A\&B$ from the states of A and B. Section 3 develops this argument for the internal indistinguishability of the Bell states.

In Section 4 we discuss similarities and dissimilarities to Gödel's Theorem and the Halting Problem.

2 Restrictions on Measurements from Inside

In this section we review an argument [3] why it is impossible for an observer to distinguish all states of a system in which she or he is contained. The argument exploits self-reference properties, and it does not make any assumptions about the character of the time evolution.

Description of measurements Let us assume that we have a physical theory whose formalism specifies for the systems it describes sets of possible states. In general these states may describe classical or quantum systems, and they may refer to individual systems or to statistical ensembles. We can think of the state set to consist of the density matrices and interpret a density matrix as representing the state of an individual quantum system.

A measurement performed by an apparatus A on some observed system O establishes certain relations between the states of A and of O. After a measurement, we infer information about the state of O from information we have about the state of A. We assume the states of A and of O refer to the *same* time after the experiment. To describe this inference, let us use a map I from the power set of the set $\mathcal{S}_A$ of apparatus states into the power set of the set $\mathcal{S}_O$ of system states. (Note that the curly $\mathcal{S}_A$ denotes the set of all apparatus states, whereas we use S_A for other sets of apparatus states.) The inference map I characterises the kind of measurement performed. I is defined as assigning to every set S_A of apparatus states (except the empty set) the set $I(S_A)$ of object states compatible with the information that the apparatus after the experiment is in one of the states in S_A.

This defines the inference map I which depends on the kind of measurement we are performing. I is different in different measurement situations. But when the observer chooses the experimental set-up, she also chooses a map I describing how she is going to interpret the pointer reading after the experiment. This map is fixed throughout the measurement. The states in $I(\mathcal{S}_A)$ are the possible states of O after the experiment: They are the states of O compatible with the information that we do not know anything about the final state of A except that the measuremnt has taken place. In general not every state of O is compatible with the information that the measurement has taken place; not every state of O is a possible state after the experiment. We have $I(\mathcal{S}_A) \subsetneq \mathcal{S}_O$.

Knowing that if the apparatus after the experiment is in a state s_A, the observed system must be in a state in $I(\{s_A\})$, one infers from the information that the apparatus after the experiment is in one of the states in S_A that the state of the observed system must be in $\bigcup_{s_A \in S_A} I(\{s_A\})$. Therefore we have, according to the definition of I,

$$I(S_A) = \bigcup_{s_A \in S_A} I(\{s_A\}). \tag{3}$$

Example 1. *Assume we measure an observable represented by a self-adjoint operator M with non-degenerate eigenstates $|m\rangle$ on a Hilbert space $\mathcal{H}_O$, and assume further that the value of the pointer is represented by a self-adjoint operator P with non-degenerate eigenstates $|p\rangle$ of the apparatus Hilbert space $\mathcal{H}_A$, and assume the eigenstate-eigenvalue link: An observable represented by a self-adjoint operator has an unambiguous value if and only if the system is in an eigenstate of that operator. Under these assumptions for each eigenstate $|m\rangle$ of the measured observable there is an eigenstate $|p(m)\rangle$ of the pointer refering to it. There may also be pointer eigenstates refering to none of the eigenstates of the observable. These pointer readings indicate that the measurement failed. So the map I is*

$$I(\{|\psi\rangle\}) = \begin{cases} |m\rangle & \text{if } |\psi\rangle = |p(m)\rangle, \\ \emptyset & \text{else.} \end{cases}$$

From singleton sets we can extend I to arbitrary sets S_A of pure apparatus states by (3)*.*

Example 2. *When we measure an observable represented by a self-adjoint operator M with spectral decomposition $M = \sum_m M_m$, using an apparaturs with a self-adjoint pointer observable $P = \sum_p P_p$, a map I can be defined on the density matrices by*

$$I(\{\rho_A\}) = \begin{cases} M_m & \text{if } \rho_A = P_{p(m)} \rho_A P_{p(m)}, \\ \emptyset & \text{else.} \end{cases}$$

(Again, there may be pointer readings refering to none of the eigenstates of the measured observable. These pointer readings may indicate that the measurement failed.) From singleton sets we can extend I to arbitrary sets of apparatus states by (3)*.*

Example 3. *More generally we can describe measurements by a collection $\{M_m\}$ of measurement operators which are not necessarily projections but* *satisfy the completeness relation* $\sum_m M_m^\dagger M_m = I$*. If the state of the quantum system* O *is* ρ *immediately before the measurement then the probability that result* m *occurs is given by* $\mathrm{tr}(M_m^\dagger M_m \rho)$ *and if this result occurs the state of the system after the measurement is* $M_m \rho M_m^\dagger / \mathrm{tr}(M_m^\dagger M_m \rho)$*. The pointer is described by a collection* $\{P_p\}$ *of operators also satisfying the complenetess relation. A map* I *can be defined on the density matrices by*

$$I(\rho_A) = \begin{cases} M_m \rho M_m^\dagger / \mathrm{tr}(M_m^\dagger M_m \rho) & \text{if } \rho_A = P_{p(m)} \rho_0 P_{p(m)}^\dagger / \mathrm{tr}(P_{p(m)}^\dagger P_{p(m)} \rho_0), \\ \emptyset & \text{else,} \end{cases}$$

where ρ_0 *is the initial ready state of the apparatus. Again, from singleton sets we can extend* I *to arbitrary sets of apparatus density matrices states by* (3).

Discrimination of states We will say that an experiment with inference map I is able to *discriminate between the states* s_1, s_2 of O if there is one set S_A^1 of final apparatus states referring to s_1 but not to s_2, and another set S_A^2 referring to s_2 but not to s_1: $I(S_A^1) \ni s_1 \notin I(S_A^2)$ and $I(S_A^1) \not\ni s_2 \in I(S_A^2)$.

Example 4. *Consider a measurement performed by a qubit* A *on a two qubit system* $O = B\&C$*, as in Fig. 1. The state space* $\mathcal{S}_A$ *is the Bloch sphere, the state space* $\mathcal{S}_O$ *equals* $\{\alpha \in C^4 : |\alpha| = 1\}$*. Since* $\mathcal{S}_A$ *and* $\mathcal{S}_O$ *are of the same cardinality there is a bijection between them. Any such bijection gives rise to an inference map* I *which discriminates all states of* O*.*

Example 4 illustrates two important conceptual points. First, it shows that a smaller system can discriminate all states of an external system with more degrees of freedom, if the observed system is external to the apparatus. Size is not the reason for the limitations on measurements from inside. The restrictions on state discrimination *from inside* are not due to the apparatus being “smaller” than the observed system. Second, the external system A can access the non-local information encoded in the entanglement between B and C. The unobservabilty of entanglement from inside is a result following from self-reference [4]; it is not an assumption entering into our description of measurements.

Measurements from inside Now consider the case where the apparatus is measuring a system in which it is contained (see Fig. 2). So the

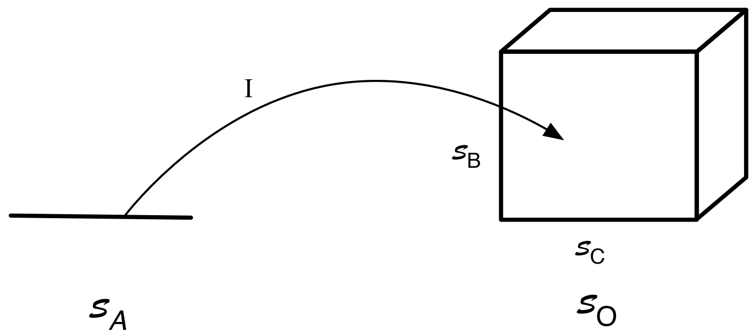

Figure 1: **All states can be discriminated in measurements from outside.** The Bloch sphere representing the state space of A is depicted as an interval $\mathcal{S}_A$. The state space of $O = B\&C$ is represented by the cube $\mathcal{S}_O$, which contains the state spaces $\mathcal{S}_B$ and $\mathcal{S}_C$, as well as another dimension representing the entanglement between B and C. Any bijection between the interval $\mathcal{S}_A$ and the cube $\mathcal{S}_O$ is an inference map discriminating all states of O.

observed system O is composed of the apparatus A and of a residue B, $O = A\&B$. We assume that the observed system has strictly more degrees of freedom than the apparatus and contains it. This can be formulated in an *assumption of proper inclusion*:

$$(\exists \rho, \rho' \in \mathcal{S}_O) : R_A(\rho) = R_A(\rho'), \rho \neq \rho'.$$

Here $R_A(\rho)$ is the partial trace over B of O's density matrix ρ. It is the state of A determined by restricting the state ρ of O to the subsystem A.

Whether the assumption of proper inclusion is satisfied or not depends not only on the sets $\mathcal{S}_A, \mathcal{S}_O$ but also on the restriction map R_A. ([3] gives an example of two restriction maps such that the assumption of proper inclusion is satisfied with respect to one but not the other.) This may seem odd but it is not. An arbitrary subset of $\mathcal{S}_O$ can in general not be interpreted to be the set of states of a subsystem of B. The restriction map R_A gives physical information which is not reflected in the structure of the sets $\mathcal{S}_A$ or $\mathcal{S}_O$, namely the fact that A is a subsystem of O. That A is a subsystem of O does not only depend on the abstract structure of A (and of O), but on *which* system A is. If A and A' are isomorphic and A is a subsystem of O, it does not follow that A' is a subsystem of O.

The assumption of proper inclusion seems trivial in the sense that the bigger system O needs more parameters to fix its state. But it excludes situations in which each physically possible state of O is uniquely determined by the state of a subsystem A together with some constraint. (We take constraint to mean that states violating the constraint are physically impossible in the sense that the system can never be in such a state. Think for example of O as consisting of A and a mirror B reflecting exactly the state of A; in this case the assumption of proper inclusion is not fulfilled although A is a subsystem of O.)

A consistency condition The states of the apparatus after the measurement are self-referential: they are states in their own right, but they also refer to states of the observed system in which they are contained. This leads to a *consistency condition* for the inference map I which must be satisfied lest the inference map be contradictory: For every apparatus state ρ_A, the restriction of the system states $I(\{\rho_A\})$ to which it refers should again be the same apparatus state ρ_A. This can be written as:

$$R_A(I(\{\rho_A\}) = \{\rho_A\}.$$

for all possible post-measurement states ρ_A of the apparatus. This consistency condition is illustrated in Fig. 2.

From the physical point of view the consistency condition is not a restrictive requirement. Rather it is motivated by logic: It ensures that we cannot arrive at contradictory conclusions about the apparatus state. Assume that the meshing condition is violated and that therefore there is a state $\rho' \in I(\{\rho_A\})$ such that $R_A(\rho') \neq \rho_A$. Then, knowing that after the experiment the apparatus is in the state ρ_A, we would conclude that O is in one of the states in $I(\{\rho_A\})$, possibly in ρ'. From this in turn we conclude that A can be in the state $R_A(\rho')$, which contradicts the assumption that A is in the state ρ_A. Note that the consistency condition has to be imposed because both ρ_A and $R_A(I(\{\rho_A\}))$ describe the state of A at the same time.

Restricted state discrimination from inside The consistency condition and the assumption of proper inclusion imply restrictions on the distinguishability of states from inside. *For an apparatus A contained in O there is no inference map I, and thus no experiment, which can distinguish states of O whose restrictions to A coincide.* More precisely, if ρ_1, ρ_2 are two states

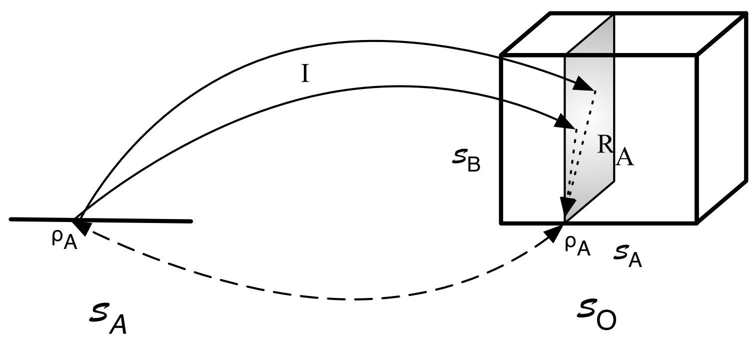

Figure 2: **Not all states can be discriminated in measurements from inside.** The state space of A is represented by an interval $\mathcal{S}_A$, the state space of $O = A\&B$ by the cube $\mathcal{S}_O$, which contains the state spaces $\mathcal{S}_A$ and $\mathcal{S}_B$, as well as another dimension representing the entanglement between A and B. The grey plane represents the states of O whose restriction R_A to A (dotted line) coincides. The consistency condition requires of I that the projection R_A of any state in $I(\{\rho_A\})$ is again ρ_A. Therefore the bijection (represented by the dashed line) identifying the original $\mathcal{S}_A$ with the subspace $\mathcal{S}_A \subset \mathcal{S}_O$ has to be the identity map. The discrimination of two different O-states in the grey plane requires them to be in the image of two *differrent* A-states, which is prevented by the consistency condition. No bijection between the interval $\mathcal{S}_A$ and the cube $\mathcal{S}_O$ satisfies the consistency condition.

of O fulfilling $R_A(\rho_1) = R_A(\rho_2)$, then there is no inference map I, and thus no measurement using as apparatus A, which can discriminate between ρ_1 and ρ_2: For all inference maps I satisfying the consistency condition, there exist no two sets of A-states S_A^1, S_A^2 such that S_A^1 refers to ρ_1 but not to ρ_2 and that S_A^2 refers to ρ_2 but not to ρ_1. This result is proved in [3] and illustrated in Fig. 2.

A particular kind of measurement from inside is induced by measurements from outside, because every measurement of A on an external system B can be interpreted as a measurement of A on $A\&B$. One can describe the measurement of A on B by an inference map I_B: If we know that A after the measurement is in some state in S_A, we infer that B is in sone state in $I_B(S_A)$. The measurement on B also provides information about $A\&B$, which is described by the inference map

$$I_{A\&B}(S_A) := \{\rho : R_A(\rho) \in S_A, R_B(\rho) \in I_B(S_A)\}. \tag{4}$$

In this particular kind of measurement from inside the restrictions on state discrimination from inside are especially obvious. No measurement I_B from outside can induce a measurement $I_{A\&B}$ from inside which can discriminate between states ρ_1, ρ_2 whose restriction to A coincides. Any measurement I_B discriminating $R_B(\rho_1), R_B(\rho_2)$ would have to leave A in *different* states depending on whether B is in state $R_B(\rho_1)$ or $R_B(\rho_2)$. This contradics $R_A(\rho_1) = R_A(\rho_2)$.

The result on restricted state discrimination from inside does not only apply to measurements from inside induced by measurements from outside. It allows for arbitrary consistent inference maps I, which need not be of the form (4).

3 Inside Indistinguishability of the Bell states

Now let us return to quantum mechanics and the question of discriminating between the Bell states of eqns. (1) and (2). Tracing out the state of B we get

$$R_A(\rho_1) = \mathrm{tr}_B(\rho_1) = \mathbf{1}_2/2 = \mathrm{tr}_B(\rho_2) = R_A(\rho_2).$$

The reduced states $R_A(\rho_1)$ and $R_A(\rho_2)$ are no longer pure states since $\mathrm{tr}((\mathbf{1}_2/2)^2) = 1/2 < 1$. Tracing out the state of B leads to a loss of information. All this is well known.

The main result of this contribution is that *neither A nor B will be able to tell whether $A\&B$ is in the Bell state ρ_1 or ρ_2.* The proof is an application of the restriction on state discrimination from inside: Because $R_A(\rho_1) = R_A(\rho_2)$ and $\rho_1 \neq \rho_2$ the assumption of proper inclusion is satisfied. Restricted state discrimination from inside implies that there is no inference map I, and thus no measurement using as apparatus A, which can discriminate between ρ_1 and ρ_2. On the other hand we have $R_B(\rho_1) = \mathrm{tr}_A(\rho_1) = \mathbf{1}_2/2 = \mathrm{tr}_A(\rho_2) = R_B(\rho_2)$. This implies that there is no inference map I, and thus no measurement which B could perform in order to discriminate between ρ_1 and ρ_2. Neither A nor B can discriminate between the Bell states. Even if A and B exchange information about results of measurements they have performed either on themselves or on the other, neither A nor B nor the two together will not be able to tell whether $A\&B$ is in the Bell state ρ_1 or ρ_2, as we will argue below. Still, the Bell states ρ_1 and ρ_2 *are* different. An outside observer can distinguish them by measuring observables pertaining to both A and B.

It is the presence of entanglement which prevents the unique determination of the state of $A\&B$ from the states of A and B. If either A or B is classical, no entanglement is possible. This is easiest to see from Bell's inequalities in the formulation of Csirel'son [5] , which read

$$\begin{aligned} & |E(A_1B_1) + E(A_2B_1) + E(A_2B_2) - E(A_1B_2)| \\ \leq \; & \sqrt{\mathrm{tr}\left((4\mathbf{1}_4 + ([A_1, A_2][B_1, B_2]))\rho\right)} \\ \leq \; & 2\sqrt{2} \end{aligned}$$

for all density matrices ρ and all observables A_1, A_2 with eigenvalues ± 1 of system A and likewise B_1, B_2 of system B. For the density matrices of the Bell states the inequalities are satisfied with equality.

System A is classical if all observables of A commute, i.e. if $[A_1, A_2] = 0$ for all A_1, A_2. In this case $|E(A_1B_1) + E(A_2B_1) + E(A_2B_2) - E(A_1B_2)| = 2$ for all ρ, irrespective of whether system B is classical or quantum. No entanglement is possible. The same holds if B is classical. The density matrices ρ_A of A and ρ_B of B determine uniquely the state $\rho_A \otimes \rho_B$ of $A\&B$. This is the *only* state whose restriction to A is ρ_A and to B is ρ_B. Therefore, if A or B or both are classical all states which differ can be discriminated by A and B in cooperation.

But if both A and B are quantum systems, there will be observables for which the Bell inequalities are violated. A can at best determine the state ρ_B of B, and B can at best determine the state ρ_A of A. When they communicate these measurement results to each other, they both know that $A\&B$ is in some state ρ with $R_A(\rho) = \rho_A$ and $R_B(\rho) = \rho_B$. But this does not determine the state of $A\&B$ uniquely. For example, the Bell states agree in both their restrictions to A and B. Therefore, if both A or B are quantum systems, there are states which cannot be discriminated by A and B, even if they cooperate.

The argument is illustrated by Fig. 2. States distinguishable by A are in the vertical grey plane. States distinguishable by B are in a horizontal plane. By communicating their mesaurement results A and B learn that the state of $A\&B$ is in the intersection of the two planes. Since this intersection is not a single point, A and B cannot not determine a unique state.

The indistinguishability of certain states *for each inside observer*, is specific to quantum mechanics, although restricted state discrimination from inside holds in classical physics as well. In the classical realm, for any two states there will always be some inside oberver able to discriminate between the two. But in quantum mechanics there are states like the Bell states, between which neither A nor B will be able to discriminate. In this sense quantum mechanical entanglement aggravates the self-reference problems of measurements from inside.

4 Similarities and Dissimilarities to Gödel's Theorem and the Halting Problem

Rosser [6] pointed out that the techniques used in the theorems of Gödel [7], Church [8], and Turing [9] are very similar. Kleene [10] showed Gödel's incompleteness theorem to follow from the insolubility of the halting problem. We conclude by pointing out some parallels and some differences between the above results on restricted state discrimination from inside and the theorems of Gödel, Church, and Turing.

The similarities can be summarised by the three pairs propositions–states, proof–measurement, and effectively generated proof–measurement from inside. (1) Statements about a physical system are statements about the state of the system. Propositions in formal systems and results of programmes correspond to physical states. (2) A measurement on a physical system establishes the truth or falsehood of some statements about the system. In this spirit, the state of a system is often regarded as a full or partial assignment of truth values. Both proofs and measurements are decision procedures: A proof establishes the truth or falsehood of statements in a formal system and corresponds to a measurement on a physical system. (3) A measurement from inside is self-referential, because its result has implications for the state of the observer. The arithmitisation of syntax achieved by the Gödel numbering, together with the restriction to effectively generated proofs, allows for the formulation of self-referential statements which are undecidable.

But there also important dissimilarities: First, the consistency condition for measurements from inside is different from the assumption of ω-consistency in Gödel [7] resp. consistency in Rosser [11]. Second, both quantum mechanics and classical mechanics use continuous state spaces, while in the theorems of Gödel, Church, and Turing propositions and programmes are referenced by natural numbers. The importance of this difference remains to be clarified. Perhaps it is not as important as it seems, since indistinguishability for measurements from inside holds for both discrete and continous sets of states. Third, indistinguishability for measurements from inside requires no substantial assumption about richness of state set other than the assumption of proper inclusion. In the theorems of Gödel, Church, and Turing the asumption about the formal system being rich enough for natural numbers is essential.

Turing [12] famously pointed to the fallacy of assuming that everything that can be known in principle is known actually and immediately. The difference of the two became the subject of complexity theory. The inside indistinguishability of the Bell states is a matter of principle, as are the theorems of Gödel, Church, and Turing. We have no insight to offer about the efficient distinguishability of states, except that the in principle indistinguishability of the Bell states from inside implies efficient indistinguishability from inside. The Bell states are key in quantum teleportation, as well as in quantum algorithms involving the Hadamard gate or Fourier transforms, like the Deutsch-Jozsa [13] algorithm. In this field the implications of the inside indistinguishability of the Bell states remain to be seen.

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