Stress Testing: From Arts to Science

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Outline

Purpose of Stress Testing: Complement statistical risk measurement

- Stress Tests: Which scenarios lead to big losses?
  Derive risk reducing action.
  (Statistical risk measurements: What are prob's of big losses?)
- Stress Tests: Address model risk.
  Consider alternative risk factor distribution.
  (Statistical risk measurement: Assume fixed model.)

Requirements on stress scenarios (Basel II)

- plausible
- severe
- suggestive of risk reducing action

See Basel Principles of Sound Stress Testing

Framework

Reference risk factor distribution ν,
Portfolio loss function L,
both on risk factor space $\Omega \subset \mathbb{R}^n$.

First Generation Stress Tests: Hand-picked Point Scenarios

- Point scenario: each risk factor gets a value: $\mathbf{r} \in \Omega$
- A small number of scenarios is picked by hand, ideally involving heterogeneous groups of experts.

$$A = \{\mathbf{r}^1, \mathbf{r}^2, \ldots, \} \subset \Omega$$

a small set of hand-picked scenarios.

- Find worst case scenario and worst case loss in A

$$\max_{\mathbf{r} \in A} L(\mathbf{r})$$

- Worst case loss over A is a coherent risk measure.

First Generation Stress Tests: Examples

- most stress tests of market or credit risk performed by financial institutions
- SPAN rules
- FSAP stress tests
- institutional stress tests during 2009 crisis

Criticism of First Generation Stress Tests

Are there any real stress tests
whose results forced a bank to change strategy?

Accidental or deliberate misrepresentation of risks:

1. Neglecting severe but plausible scenarios
2. Considering too implausible scenarios

Outline

Second Generation Stress Tests: Plausible Scenarios

- Measure of plausibility for point scenarios:

$$\mathrm{Maha}(\mathbf{r}) := \sqrt{(\mathbf{r} - \mathbb{E}(\mathbf{r}))^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - \mathbb{E}(\mathbf{r}))},$$

  where Σ is covariance matrix of risk factor distribution ν.
- Intuition:
  Scenarios in which some risk factors move **many standard deviations** are implausible.
  Scenarios in which some pair of risk factors moves **against their correlation** are implausible.

Second Generation Stress Tests: Systematic Point Scenario Analysis

- Set of plausible scenarios

$$A := \mathrm{Ell}_h := \{\mathbf{r} : \mathrm{Maha}(\mathbf{r}) \le h\},$$

  where h is the plausibility threshold.
- Systematic search of worst case scenario:

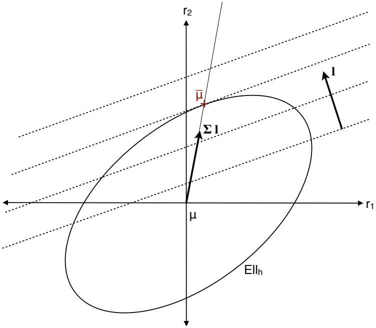
$$\max_{\mathbf{r} \in \mathrm{Ell}_h} L(\mathbf{r})$$

Second generation stress of linear portfolio

- Loss linear function of n normal risk factors:
  $L(\mathbf{r}) = \mathbf{l}^T(\boldsymbol{\mu} - \mathbf{r}), \nu \sim N(\boldsymbol{\mu}, \Sigma)$.
- Systematic search of worst case scenario:

$$\max_{\mathbf{r} \in \mathrm{Ell}_h} \mathbf{l}^T(\boldsymbol{\mu} - \mathbf{r}).$$

- Worst case scenario: $\overline{\boldsymbol{\mu}} = \boldsymbol{\mu} - \frac{h}{\sqrt{\mathbf{l}^T \Sigma \mathbf{l}}} \Sigma \mathbf{l}$
- Worst case loss: $\mathbb{E}_{\overline{Q}}(L) = h\sqrt{\mathbf{l}^T \Sigma \mathbf{l}}$.



Advantages of Systematic Stress Testing with Point Scenarios

All three requirements on stress testing are met:

- Do not miss plausible but severe scenarios.
- Do not consider scenarios which are too implausible.
- Worst case scenario over Ell_h gives information about portfolio structure and suggests risk reducing action.

Problems of Systematic Stress Testing with Point Scenarios

1. What if risk factor distributions ν is non-elliptical?
2. What if risk true factor distribution is not ν?
   Model risk is not addressed.

3. Maha does not take into account fatness of tails.
4. $\mathrm{MaxLoss}_{\mathrm{Ell}_k}$ depends on choice of coordinates.

Outline

Mixed Scenarios

Mixed scenario: Probability distribution of point scenarios.

- Interpretation 1:
  Risk factor distributions alternative to the prior ν.
  Model risk.
- Interpretation 2:
  Generalisation of point scenarios,
  but support not concentrated on one point.

Plausibility of Mixed Scenarios

- **Measure of plausibility for mixed scenarios:**
  relative entropy from ν
  (I-divergence, information gain, Kullback-Leibler distance)

$$I(Q||\nu) := \begin{cases} \int \frac{dQ}{d\nu}(\mathbf{r}) \log \frac{dQ}{d\nu}(\mathbf{r}) d\nu(\mathbf{r}) & \text{if } Q \ll \nu \\ +\infty & \text{if } Q \not\ll \nu \end{cases}$$

- Intuition:
  Relative entropy $I(Q||\nu)$ measures the 'distance' of the distributions Q and ν.
  $I(Q||\nu) = 0$ if and only if $Q = \nu$ (as distributions)

Worst Case Scenario

- **Set of plausible scenarios:** Instead of ellipsoid take Kullback-Leibler sphere in the space of distributions

$$A := S(\nu, k) := \{Q : I(Q||\nu) \leq k\}.$$

- **Severity of scenarios:** Instead of $L(\mathbf{r})$ take $\mathbb{E}_Q(L)$
- Systematic stress test with mixed scenarios:

$$\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L) =: \mathrm{MaxLoss}_k(L)$$

If it exists, call scenario achieving MaxLoss: $\overline{Q}$.

Advantages of Systematic Stress Testing with Mixed Scenarios

1. Scenario set is naturally defined for non-elliptical risk factor distributions ν.
2. Model risk is addressed:
   Mixed scenarios are alternatives to prior risk factor distribution ν.
3. Relative entropy does take into account fatness of tails of ν.
4. $\mathrm{MaxLoss}_k$ does not depend on choice of coordinates.

Outline

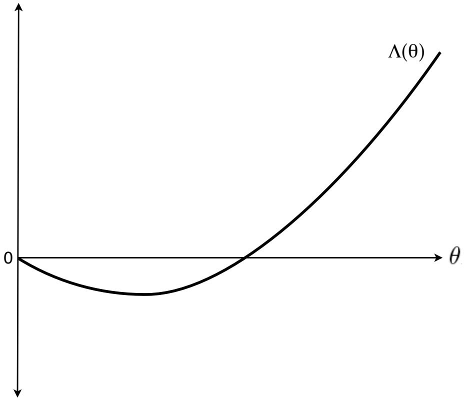
The Basic Tool

- Tool from large deviations theory for solving explicitly the optimisation problem $\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L)$:

$$\Lambda(\theta) := \log \left(\int e^{\theta L(\mathbf{r})} d\nu(\mathbf{r}) \right).$$

Basic Properties of the Λ-function

- $\Lambda(0) = 0$.
- Λ is convex.



Solution of Worst Case: The Generic Case

Theorem

- *Except in the pathological cases (i), (ii), (iii) below, the equation*

$$\theta\Lambda'(\theta) - \Lambda(\theta) = k, \tag{1}$$

  has always a unique positive solution $\overline{\theta}$.

- *The mixed worst case scenario* $\overline{Q}$ *is the distribution with* ν*-density*

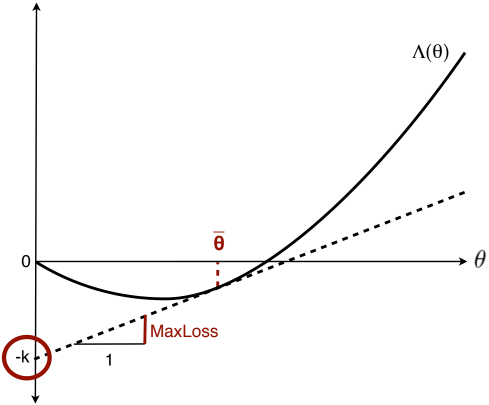
$$\frac{d\overline{Q}}{d\nu}(\mathbf{r}) = e^{\overline{\theta}L(\mathbf{r})-\Lambda(\overline{\theta})}, \tag{2}$$

- *The Maximum Loss achieved in the mixed worst case scenario* $\overline{Q}$ *is*

$$\mathbb{E}_{\overline{Q}}(L) = \Lambda'(\overline{\theta}).$$

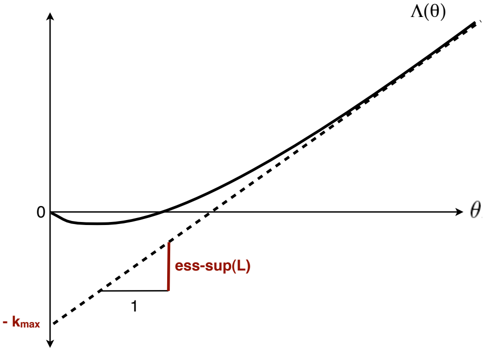
Practical Calculation of Worst Case

1. Calculate $\Lambda(\theta)$. (Evaluate n-dimensional integral.)
2. Starting from the point $(0, -k)$, lay a tangent to $\Lambda(\theta)$ curve.
3. Worst case loss is given by the slope of the tangent.
4. Worst case scenario is distribution with density $\frac{d\overline{Q}}{d\nu}(\mathbf{r}) = e^{\overline{\theta}L(\mathbf{r}) - \Lambda(\overline{\theta})}$, where $\overline{\theta}$ is θ-coordinate of tangent point.



Worst Case: The Pathological Cases

(i) If $\operatorname{ess\,sup}(L)$ is finite, and $k \geq -\log(\nu(\{\mathbf{r} : L(\mathbf{r}) = \operatorname{ess\,sup}(L)\}))$, then $\mathrm{MaxLoss}(k) = \operatorname{ess\,sup}(L)$.

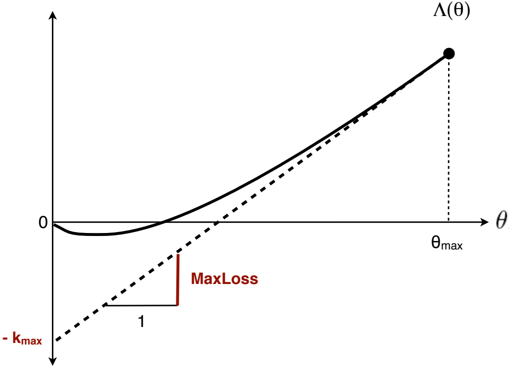


(ii) If $\theta_{\max} = 0$ then $\mathrm{MaxLoss}(k) = \infty$ for all $k > 0$.

Worst Case: Pathological Cases cont'd

$\theta_{\max} := \sup\{\theta : \Lambda(\theta) < +\infty\}$: maximal θ-value for which Λ is finite.

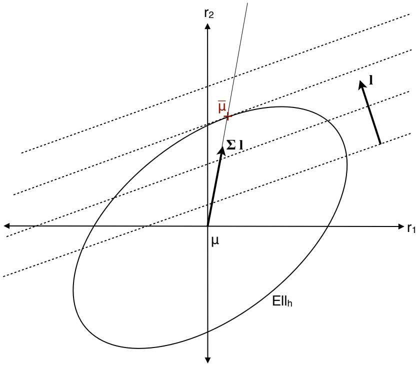
(iii) If $\theta_{\max}, \Lambda(\theta_{\max})$, and $\Lambda'(\theta_{\max})$ are all finite and $k > k_{\max} := \theta_{\max}\Lambda'(\theta_{\max}) - \Lambda(\theta_{\max})$, then $\mathrm{MaxLoss}(k) = (k + \Lambda(\theta_{\max}))/\theta_{\max}$.



Outline

Normal risk factors, linear portfolio

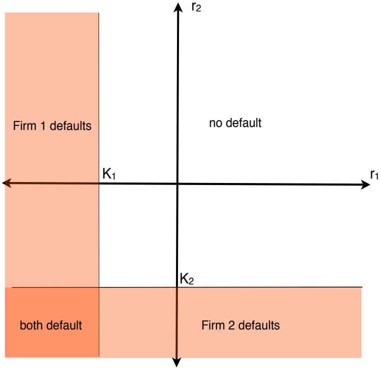
- Loss linear function of n risk factors: $L(r) = \mathbf{l}^T(\mu - r), \nu \sim N(\boldsymbol{\mu}, \Sigma)$.
- Λ quadratic: $\Lambda(\theta) = \mathbf{l}^T \Sigma \mathbf{l} \theta^2 / 2$.
- Worst case scenario: $\overline{Q} \sim N(\overline{\boldsymbol{\mu}}, \Sigma)$ with $\overline{\boldsymbol{\mu}} = \boldsymbol{\mu} - \frac{h}{\sqrt{\mathbf{l}^T \Sigma \mathbf{l}}} \Sigma \mathbf{l}$ where $h = \sqrt{2k}$.
- Worst case loss: $\mathbb{E}_{\overline{Q}}(L) = h\sqrt{\mathbf{l}^T \Sigma \mathbf{l}}$.



Stressed default correlations

Simple firm value model of portfolio credit risk

- ν: firm values $r_i \sim N(0,1)$, correlation ρ
- Default barrier K_i for Firm i
- Joint default probability $p_{12} := \Phi_{2,\rho}(K_1, K_2)$
- Loss l_i if Firm i defaults
- $L(\mathbf{r}) = \sum_{i=1}^{n} l_i 1_{(-\infty, K_i]}(r_i)$



Stressed default correlations

- Λ-function

$$\Lambda(\theta) = \log\Big((p_1 - p_{12})e^{\theta l_1} + (p_2 - p_{12})e^{\theta l_2} + p_{12}e^{\theta(l_1+l_2)} + (1 - p_1 - p_2 + p_{12})\Big).$$

- Maximum Loss

$$\frac{(p_1 - p_{12})l_1 e^{\overline{\theta} l_1} + (p_2 - p_{12})l_2 e^{\overline{\theta} l_2} + p_{12}(l_1 + l_2)e^{\overline{\theta}(l_1+l_2)}}{(p_1 - p_{12})e^{\overline{\theta} l_1} + (p_2 - p_{12})e^{\overline{\theta} l_2} + p_{12}e^{\overline{\theta}(l_1+l_2)} + (1 - p_1 - p_2 + p_{12})}$$

- Worst case default prob's

$$\begin{aligned}
\overline{p}_{12} &= \exp[\overline{\theta}(l_1 + l_2) - \Lambda(\overline{\theta})]p_{12} \\
\overline{p}_1 - \overline{p}_{12} &= \exp[\overline{\theta} l_1 - \Lambda(\overline{\theta})]p_1 - \overline{p}_{12} \\
\overline{p}_2 - \overline{p}_{12} &= \exp[\overline{\theta} l_2 - \Lambda(\overline{\theta})]p_2 - \overline{p}_{12} \\
\overline{p}_0 &= 1 - \overline{p}_1 - \overline{p}_2 + \overline{p}_{12}
\end{aligned}$$

Stressed default correlations

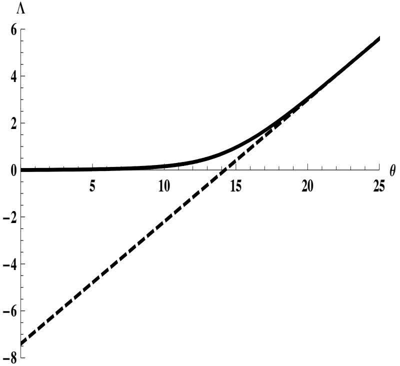
Numerical example

- def prob's: $p_1 = 1.33\%$, $p_2 = 0.02\%$
- asset correlation $\rho = 0.5$
- LGD: $l_1 = 0.5$ and $l_2 = 0.4$
- plausibility threshold $k = 2$

	no def	F1 def	F2 def	both def	def. corr.	exp. loss
est'd dist. ν	98.66%	1.32%	0.013%	0.007%	4.23%	0.67%
worst c. dist.	43.14%	47.94%	0.19%	8.36%	26.15%	32.01%

Stressed transition probabilities

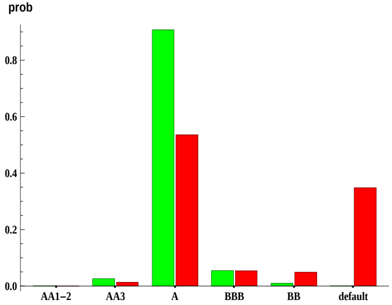
- $\Omega = \{0, 1, \dots n\}$: rating classes.
- $\mathbf{p} = (p_1, \dots, p_n)$: estimated transition probabilities
- $\mathbf{l} = (l_1, \dots, l_n)$: loss caused by transitions
- $\Lambda(\theta) = \log\left(\sum_{j=1}^{n} p_j \exp(\theta l_j)\right)$.
- Worst case transition probabilities: $\overline{p}_i = \frac{p_i \exp(\overline{\theta} l_i)}{\sum_{j=1}^{n} p_j \exp(\overline{\theta} l_j)}$.



Numerical example: A-rated bond

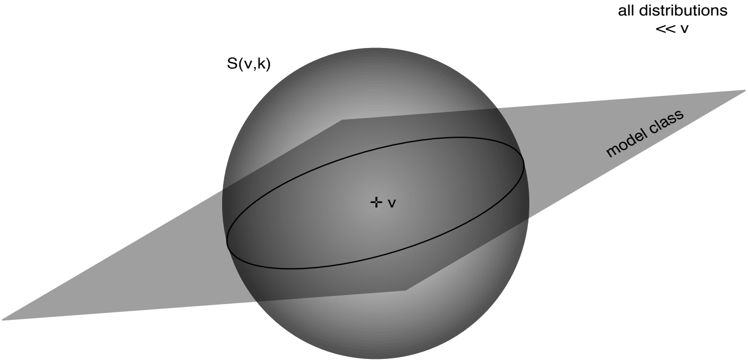
	AA1-2	AA3	A	BBB	BB	Default
loss from transitions [%]	-3.20%	-1.07%	0.00%	3.75%	15.83%	51.80%
est'd trans. prob. [%]	0.09	2.60	90.75	5.50	1.00	0.06
worst c. trans. prob. [%]	0.036	1.34	53.53	5.37	4.91	34.8

Expected loss from transitions under est'd probs: 0.37%
Expected loss from transitions under worst case probs at k=2: 19.07%



Model Risk

- estimation error: wrong distribution parameters
- model misspecification: wrong model class
- $\sup_{Q \in S(\nu,k)} \mathbb{E}_Q(L)$ quantifies effects of both on expected loss.



Summary

Systematic stress tests with mixed scenarios

- do not neglect dangerous scenarios when they are plausible,
- do not produce highly implausible scenarios,
- are applicable to both continuous and discrete risk factors with arbitrary distributions,
- quantify the effects of model risk...

... and can be implemented straighforwardly.

Link

http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1328022