Endogenous leverage and asset pricing in double auctions

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A B S T R A C T
We propose a double auction mechanism for the exchange of leveraged assets and bonds in an agent based model. In this framework we validate recent results in general equilibrium theory about endogenous leverage and its consequences for asset pricing. We find that the institutional details of exchange are critical for a good match between the theoretical equilibrium state and the final state of the double auction: Specifically, the outcome of the double auction is sensitive to the details of how markets for debt and collateral are coordinated and how collateral is cleared. When trade is restricted to neighbours in a network, final prices and allocations are significantly different from unrestricted equilibrium.

1. Introduction

We study the exchange and pricing of leveraged assets in an agent based model of a continuous double auction. Specifically we want to understand how the leverage that can be achieved in the market is determined and how leverage affects the prices of assets. Understanding these issues is key to the understanding of financial crisis more generally because leverage makes agents more sensitive to changes in asset prices. The recent literature on the financial crisis identifies leverage as one of the key drivers of systemic risk (see Adrian and Shin, 2010; Geanakoplos, 2010; Shin, 2010 and Gorton and Metrick, 2012 among others).

While in many of the classical references in the literature on collateralized credit such as Kiyotaki and Moore (1997) or Bernanke et al. (1996) leverage is an exogenous parameter, Geanakoplos and Zame (2013) and Geanakoplos (2010) recently proposed ideas that allow endogenizing leverage within a general equilibrium framework. These ideas rely on a pure mechanism of supply and demand with quantity constraints. In this theory the amount that can be borrowed against a
particular asset to purchase it is determined in the market. Endogenous leverage also has asset pricing implications: In general, both equilibrium asset prices and the prices of debt are distorted. We study endogenous leverage and asset pricing via double auctions to validate this theory. Continuous double auctions are competitive trading mechanisms which are used in practice at various organized exchanges. In the economics literature double auctions have been studied in game theory (see Wilson, 1987; Easley and Ledyard, 1997; Mertens, 2003; Giraud, 2010) but mainly in experimental economics (see Smith, 2008; Sunder, 1995; Bossaerts, 2002, 2009). Experimental economists believe that the continuous double auction is a trading institution that comes close to an environment which abstract equilibrium theories of competitive trading try to describe. It is an institution that allows for competitive bidding and trade on both sides of the market over time. One of the discoveries of experimental economists is that in many experiments double auctions converge to states where trading activity comes to a halt. In these final states prices and allocations often are similar to what equilibrium theory predicts.

We build on this insight and ask whether the competitive theory of trade in leveraged assets has descriptive and predictive power in a double auction environment. Rather than running an experiment with human subjects, we use an agent based model of a continuous double auction in which the agents are modeled by zero intelligence agents. These are agents who always make bids which improve their utility but otherwise bid randomly.

In analogy to methods in experimental economics, studying a simulated double auction serves two purposes: First, it specifies the institutions and processes behind the abstract equilibrium theory of leveraged assets. To find out how the theory applies to specific institutions we perform a simulation study with double auctions. Second, we believe that the simulated double auction helps us to gain a deeper understanding of the theory, its scope and limitations.

Our paper makes the following three main contributions: First, while double auctions for competitive asset trading have been studied for a long time, the analysis of double auctions for leveraged assets is new. This extension is not straightforward: In particular, the extended institution has to specify how collateral is cleared and how leveraged purchases are coordinated across asset and debt markets. We propose such specifications and implement them in our simulation. Second, our main empirical finding is that the details of the institutional specification matter a lot for how well the final state of the double auction agrees with the equilibrium. Which details exactly matter in practice seems to be not obvious. We study three institutions of exchange which all have the necessary properties that allow the double auction to approach the theoretical equilibrium. But the final state matches the equilibrium only for one of the three institutions. In the two other institutions there is no convergence of prices in the double auction, or if there is convergence it is to a state significantly different from equilibrium. For an institution to work it is not sufficient to offer transaction channels for assets and bonds to be transferred between arbitrary agents. These channels are only used effectively when the transactions they allow are advantageous to both partners, if they are not limited by unnecessary constraints, and if they are in line with the information and the strategy of the partners. Third, we analyze where the endogenous leverage model reaches its limits for predicting simulation results. Our model with zero intelligence agents puts the emphasis on the role of institutions for market outcomes with little impact from the intelligence of agents or the evolution of sophisticated market strategies. Final prices and allocations emerging in the double auction are robust to the preferences of the agents, to the number of agents, and to their endowment. But when trade is restricted to neighbours in a given network, final prices and allocations may be significantly different from unrestricted equilibrium.

Overall we believe that our results provide evidence that collateral equilibrium theory is a valid and appropriate framework for studying trade in collateralized assets if (i) the assets provide no direct utility (like various securities for example), (ii) debt instruments are traded on competitive markets with functioning institutions coordinating collateral and debt markets, (iii) when agents make suboptimal trades institutions are in place that allow for easy reversibility of trades, and (iv) there are no restrictions on trade between arbitrary agents. Repo markets and certain securitization markets would for example be markets for which the theory provides a suitable frame for analysis.

2. The theoretical benchmark

The general equilibrium model of leverage and asset pricing we study in this paper via agent based models of double auctions is due to Geanakoplos and Zame (2013). There are two key properties of collateral equilibrium emphasized by the theory:

1. Contract selection: In equilibrium not all debt instruments available in the market will be traded.
2. Collateral premium: In equilibrium assets that can be used as collateral for debt instruments trade at a premium above their fundamental value (the present value of the dividend stream provided by these assets).

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2 Endogenous leverage and asset pricing was also studied in the corporate finance literature. The classical reference in this literature is Holmstrom and Tirole (1997). However this literature considers situations of asymmetric information and incentives, where leverage is bounded by incentives that lenders need to provide to borrowers to improve (or not to opportunistically manipulate) the dividends or values of assets. This is an entirely different mechanism from the mechanism studied in the general equilibrium theory of endogenous leverage. While this is certainly an important mechanism, the holders of mortgage securities, the owners and the holders of securities used in repos – asset classes that played a key role in the recent financial crisis – had no control over the values of these securities or over their dividends. Another strand of the literature in which endogenous leverage has been studied is the literature on agent based models of financial markets. See for instance Friedman and Abraham (2009) or Geanakoplos et al. (2012).

3 Gode and Sunder, 1993 introduced this terminology in the literature. A perhaps more precise terminology would be to characterize the behavior as myopic utility improving behavior.

4 Earlier unpublished versions of this paper were circulated 1997, 2002, 2005 and 2009.
Geanakoplos (2010) develops a simple parametric example of collateral equilibrium with these two properties, which we use as benchmark in our analysis. Collateral premium and contract selection also hold in the general version of the theory (see Geanakoplos and Zame, 2013) and do not depend on some of the more specific features of the example. The example has the advantage that it allows for an easy computation of collateral equilibrium.\footnote{Computing collateral equilibria for the general model is quite involved, see Kübler and Schmedders (2003); Kilenthong (2011) or Schommer (2013)} For easier reference we give a brief and informal exposition of the benchmark example and its key properties here. A more formal discussion including some of the details related to equilibrium computation is given in the appendix.

The real side of the economy: The example considers an exchange economy with time and uncertainty. There are two dates, today and tomorrow, and two states of the world tomorrow, called the $U(p)$ and the $D(own)$ state. There is one consumption good and a real asset. The consumption good is the numéraire and we therefore refer to it also as “cash”. The agents care only about the sum of consumption today and the expected consumption tomorrow. The real asset has itself no utility value but yields 1 unit of the consumption good in state $U$ and 0.2 units in state $D$. Agents are all endowed today with 1 unit of the consumption good and 1 real asset. There are no endowments tomorrow. Agents are uniformly arranged on a continuum $I = [0, 1]$ where agent $i \in I$ today assigns probability $i$ to the event that state $U$ will occur tomorrow.

Market structure: It is assumed that today there are competitive markets for the real asset and for a set of bonds which allow agents to transfer goods across time and states to achieve the consumption plans they find optimal given their budget and collateral constraints.

Collateral as enforcement of financial promises: By trading bonds agents can borrow and lend. When a bond is sold today promising the payment of a face value $V$ tomorrow, this transaction amounts to the bond buyer granting the bond seller a loan amount equal to the bond price $q$.

It is assumed that the only way to enforce financial promises is by holding real assets as collateral. The way this is modeled is to describe bonds as debt instruments characterized by their face value $V$ and by their collateral requirement. The collateral requirement says that for each unit borrowed through a particular bond agents have to hold one unit of the real asset as collateral.

It is assumed that the most a borrower can lose when not honoring a financial promise is the collateral. Loosing the collateral is the only consequence of default. Thus borrowers will always deliver the minimum of the promised amount and the collateral value. The lenders, knowing this, do not care about the borrower’s identity but only about the value of the collateral. The bond with face value less or equal to 0.2, the asset payoff in the $D$ state, is risk free because the amount promised is smaller or equal than the value of the collateral in all states. Bonds with face value greater 0.2 are subject to default risk, because in the $D$ state the bond holder receives only the asset value of 0.2 rather than the promised face value.

The key economic idea of modeling collateral in this way is that each debt contract is a pair consisting of face value and collateral requirement. If a contract differs from another either by face value or by collateral requirement it is economically different and thus has a different price.

Constraints for individual decisions: Agents face the following constraints: (i) They cannot go short in the real asset. (ii) They have to respect the collateral requirement. (iii) They have to respect their budget constraints: Today they cannot spend more on consumption, asset and bond purchases than the cash they have from their endowment and from the sale of assets or bonds. Tomorrow in each state they cannot spend more on consumption and repayments than their asset payoffs plus payments received from bond holdings. Note that the repayment on a bond or the payment received from a bond holding might not be the face value but rather the value of the collateral depending on which of the two is smaller.

Collateral equilibrium: A collateral equilibrium is a tuple of prices, consumption plans and planned holdings of the real asset and bonds such that each agent has maximized his utility over the choice set specified by the constraints, and the markets for goods, real assets and all available bonds clear.

Equilibrium predictions: We will now summarize the equilibrium predictions for prices of bonds and assets, and for allocations of consumption, bonds and assets. Equilibrium is qualitatively different depending on whether or not the menu of available bonds includes a riskfree bond.

Let us first come to the equilibrium where a risk free bond is available, for example the bond with a face value of 0.2. In this case the equilibrium is characterized by a critical value $r^*$, by the asset price $p$ and the price $q$ of the riskless bond with highest face value. The equilibrium values of the three parameters are determined by equations (2)–(4) in the Appendix. The values of $p, q, r^*$ for the situations that the riskfree bond with highest face value is $V = 0, 0.1, 0.2$ are reported in lines 1–3 of Table 1. The equilibrium allocation is such that $r^*$ separates two groups of agents. Agents with $0 \leq i \leq r^*$ are pessimists and hold only cash or the riskfree bond with highest face value. Agents with $r^* < i \leq 1$ are optimists, hold only assets, and are maximally short in riskfree bonds of the highest face value, as far as allowed by the collateral constraint. The equilibrium allocation is represented in the left hand panel of Fig. 1. Note that in this equilibrium pessimistic agents are indifferent between cash and the riskfree bond. Thus the equilibrium allocation drawn in Fig. 1 is not unique.

The collateral equilibrium has both, the features of contract selection and a collateral premium for the real asset: In equilibrium agents choose to borrow and lend only by using the riskfree bond with highest face value, even if other bonds are available. This is the contract selection or endogenous leverage property of the equilibrium. An intuition for this outcome is that the bond with face value 0.2 makes the most efficient use of the scarce collateral. Would an asset buyer finance his purchase by a risky bond he has to pay the same amount in the $D$-state as if he had used riskless debt. In the
Table 1
Predictions of the equilibrium model with infinitely many agents. If riskfree bonds are available, lines 1-3 give the parameters of the equilibrium in dependence of the face value of the riskfree bond with highest face value. If only risky bonds are available, lines 4-6 give the parameters of the equilibrium in dependence of the face value of the risky bond with lowest face value.

<table>
<thead>
<tr>
<th>Bond type</th>
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<th>Equilibrium allocation</th>
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<td>Bond price (q)</td>
</tr>
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<td></td>
<td>Marginal agent (i_1)</td>
<td>Marginal agent (i_2)</td>
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Riskfree bonds available

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<th>V</th>
<th>(p)</th>
<th>(q)</th>
<th>(i_1)</th>
<th>(i_2)</th>
<th>(i_1)</th>
<th>(i_2)</th>
<th>(i_1)</th>
<th>(i_2)</th>
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<th>(i_2)</th>
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<td>0.0</td>
<td>–</td>
<td>0.596</td>
<td>1.677</td>
<td>–</td>
<td>–</td>
<td>2.477</td>
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<td>0.1</td>
<td>0.714</td>
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<td>–</td>
<td>0.642</td>
<td>1.558</td>
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<td>2.793</td>
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<tr>
<td>0.2</td>
<td>0.749</td>
<td>0.2</td>
<td>–</td>
<td>0.686</td>
<td>1.749</td>
<td>–</td>
<td>–</td>
<td>3.186</td>
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Only risky bonds available

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<th>(p)</th>
<th>(q)</th>
<th>(i_1)</th>
<th>(i_2)</th>
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Fig. 1. Equilibrium allocation when one riskfree bond \(V^1 = 0.2\) (left) or one risky bond \(V^2 = 0.5\) (right) is available. Note that with a riskfree bond (left) the equilibrium allocation is not unique, since agents with \(i \geq i^*\) are indifferent between cash and the riskfree bond.

U-state – where he values the good most – he has to pay more. The asset sellers are pessimistic and believe that there is a high chance that the D-state will occur. They do not want to make risky loans even at very low prices (high interest rates) because they assume that there is a substantial default risk if they do so. Leverage, the percentage of the value of the real asset that can be borrowed to purchase it, is determined by contract selection through the market. Leverage is endogenous.

The collateral premium arises because there is a shadow value attached to the (binding) collateral constraint. It is the difference between the asset price of 0.677 when no bonds are available and the asset price in the presence of bonds, as appearing in column 2 of Table 1.

When riskless debt is not available the equilibrium prediction changes qualitatively. The agent populations separates into three subgroups. In this case the equilibrium is characterized by two critical values \(i_1, i_2\), by the asset price \(p\) and the price \(q\) of the risky bond with lowest face value. The equilibrium values of the four parameters are determined by Eqs. (7)-(10) in the Appendix. The values of \(p, q, i_1, i_2\) for the situations that the risky bond with lowest face value is \(V = 0.3, 0.4, 0.5\) are reported in lines 4-6 of Table 1. The equilibrium allocation is such that \(i_1, i_2\) separate three groups of agents. Agents with \(0 \leq i \leq i_1\) are pessimists and hold only cash. Medium agents with \(i_1 \leq i \leq i_2\) hold only the risky bond of lowest face value. Agents with \(i_2 < i \leq 1\) are optimists, hold only assets, and are maximally short in risky bonds of the lowest face value, as far as allowed by the collateral constraint. The equilibrium allocation is represented in the right hand panel of Fig. 1.

Again, the equilibrium has both features, contract selection and a collateral premium. Among the available debt contracts the contract with lowest face value (and therefore smallest default risk) is selected. Risky bonds with higher face value have a higher default risk and are not traded although they are cheaper. Their price is determined not from trade but from condition (17). The collateral premium decreases with increasing face value of the risky bond with lowest face value, see column 2 of Table 1.

When instead of the continuum \([0, 1]\) of agents we have a finite number \(N_0\) of agents, with their degree \(i\) of optimism equally spaced on the interval \([0, 1]\), then it is more complex to determine the equilibrium. Some of the complexity is due to the fact that for finitely many agents there need not exist marginal agents \(i_1, i_2\). But determining the equilibrium numerically
is not a major difficulty. For \( N_a = 1000 \) agents the prices and allocation in equilibrium is almost indistinguishable from the equilibrium with the continuum of agents, compare Table 1 and the first Section of Table 2.

### 3. A double auction mechanism for leveraged assets

Without the possibility of borrowing and lending, Gode and Sunder (1993) studied double auction with zero-intelligence agents. Their auction mechanism is in fact identical or very similar to most of the double auction based asset pricing experiments. We add the option to trade collateralized assets. This requires a significant extension of the double auction known from the experimental literature because we have to specify institutions which allow the coordination between financial markets (where debt instruments are traded which may finance asset trades) and the market for real assets (which may serve as collateral for the debt instruments). We also need institutions for the clearing of collateral during the auction process as will be explained later in this Section. Since this is to our best knowledge the first time a double auction for the trade in collateralized assets is studied we have used specifications that seemed to us a straightforward extension of the main ideas of double auction mechanisms. The rationale behind these extensions will be given in Section 3.2.

#### 3.1. The auction mechanism

We are now going to present the main mechanism in detail.

**Markets:** The following markets are open simultaneously: asset for cash, bonds of various types for cash, assets for bonds of various types (ABM). So, if \( J \) bond types are available, there are \( 2J + 1 \) markets. In our simulation we analyze economies with one or two bond types available.

**Agents:** There is a finite number \( N_a \) of agents, labeled by \( i \in [0, 1] \). In our standard simulations we analyze economies with \( N_a = 1000 \) agents, whose \( i \) are equidistantly distributed over \([0, 1]\). (In various robustness checks we vary the number of agents and the distribution of their degrees of optimism.) The identity \( i \) of each agent describes the probability assigned to the \( U \) state. Agent \( i \) thinks that with probability \((1 - i)\) the asset will pay only \( 0.2 \) tomorrow. Agents with higher \( i \) are more optimistic.

**Bidding:** Pick an agent at random. This agent submits both buy and sell offers to the order books of all markets. All offers involving assets are for the same, fixed number of assets, which we choose to be \( 0.01 \) units of the asset. Old offers made by this agent are deleted from the order book.

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Each offer by itself has to satisfy the following constraints. (i) No short selling of assets: if an agent has zero assets, sell offers for assets (either against cash or against bonds) are not made. (ii) Budget constraint: offers to buy an asset or a bond against cash must not exceed the agent’s stock of cash. (iii) Collateral constraint (BP): offers to sell bonds or assets do not lead to a situation where the number of assets owned by the agent plus the net number of bonds owned by the agent is negative. Agents choose the price for their offers at random (uniformly) from the interval of prices which result in a utility increase (zero-intelligence agents).

**matching:** For the newly submitted orders, pick one market at random, and pick at random the buy or sell offer on this market. Check whether the new order matches the best of the old opposite orders. A buy and a sell offer match if the price of the buy offer is higher or equal than the price of the sell offer. In case of a match, (1) the transaction is performed at a price half way between the buy and the sell offer, and (2) the other offers of both agents are deleted from the order books of all markets. In case of no match, pick another market at random, and check for a match. If the offers of an agent do not find a match on any of the markets, insert its offers into the offer book, and pick another agent to make offers.

**termination:** The auction continues with bidding until no transaction was made during the last 1000 rounds.

### 3.2. Rationale behind the auction mechanism

The auction mechanism described above involves several design decisions. Any auction mechanism for trading bonds and collateralized assets has to address two basic problems:

(a) Coordination of asset and bond markets: To buy an asset one might need to issue a bond. But to issue the bond one needs to be able to pledge the asset as collateral, which one could not yet buy because it can only be bought if a bond can be issued in the first place.

(b) Reversibility of suboptimal trades: Agents submit offers which might be suboptimal, either because they do not fully anticipate the behavior of other agents, or because the identification of an optimal offer exceeds their capabilities. Thus, agents need to get out of trades made earlier in the auction. In particular, this requires to accommodate for unlocking assets already pledged as collateral. Note that reversibility is always guaranteed in auctions where assets are traded only for cash.

Our solution to problem (b) consists in allowing an agent to free a collateralized asset by buying a bond. This amounts to a collateral constraint requiring the negative net number of bonds (short minus long) to be less or equal the number of assets. In this way, the reversal of suboptimal bond transactions made earlier in the auction becomes possible: the net effect of collateral amounts to zero. We call this particular solution to trade reversibility “bond pledgeability” (BP). Without BP the collateral constraint requires agents to hold a number of assets greater or equal to the number of short bonds. Simulation results for the market mechanism without BP are shown in Table 2.

A straightforward solution to problem (a) is for agents to condition a bond sell offer on an asset buy offer. In this way, either both trades or none will be accomplished, and the agent can define amounts and prices such that both the constraints (budget and collateral) get satisfied and a utility improvement can be obtained, which could not be guaranteed in case just one of the two offers would be realized. Note that on the other side of the coupled trades there might be two different agents.

Simulation results are shown in Table 2 under the heading ‘auction without ABM’. It will turn out that with this approach, there remains, however, one subtle problem which stands at the very heart of any complex market with coupled trades in a single transaction within a double auction. Separate utility improvement in each of the coupled trades is more restrictive than a net sum utility improvement of all coupled trades. This restriction impairs convergence to the theoretical equilibrium, as we will see in the next Section.

As a remedy we define a new market for trading assets directly against bonds (ABM), thereby reducing the three agents of the coupled trades to just two, and reducing the two prices (for bond and asset) to just one (bonds for assets). This assures a comprehensive view on utility improvement for each of the buyers and sellers. All the auction mechanisms we consider fulfill the natural requirement that assets and bonds can be transferred between arbitrary agents. In the next subsection we will investigate which of these mechanisms achieves best convergence to equilibrium.

### 3.3. Simulation of the double auction

In our simulation study we examine four questions: First, do the transaction prices converge in the course of the double auction? Second, if prices converge, are the final prices close to the collateral equilibrium prices? Third, is the final allocation close to the equilibrium allocation? Fourth, does the bond contract selection (endogenous leverage) predicted by the collateral equilibrium theory also take place in the simulated double auction?

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1. This convention is different from standard conventions often made in double auctions. For the current setting with zero intelligence agents the precise way of how the gains from trade are split between agents does not matter. With strategic agents the situation would be different. We thank an anonymous referee for pointing this out.

2. This property, which could be interpreted as a netting agreement, is referred to in the literature also as pyramiding.
The outcomes of the simulated auctions are random, since transaction prices are random because of the random offers submitted by agents. This requires a statistical analysis of the final prices and allocations in the simulated auctions. To answer the first question above, we calculate a convergence indicator, given in the last column of Table 2. This convergence indicator gives the proportion of auctions in which approximate price convergence has taken place. We say we have approximate price convergence in an auction when the standard deviation of relative price fluctuations in the last 1000 transactions is less than 1%. The relative price fluctuation of a transaction is the difference between the transaction price and the average transaction price (the average being taken over all transactions among the last 1000 in the respective market), divided by the average transaction price.

To answer the second question we quantify the deviation of auction final prices from the analytical equilibrium by a price error defined as the average absolute value of the relative error of transaction prices as compared to equilibrium prices. The average is taken over the last 1000 transactions.

To answer the third question we quantify the fit of the final allocation in the auction to the equilibrium allocation by an allocation error indicator. It is defined as half the average over all agents, of the absolute value of the cash difference held at the end of the auction and in equilibrium, plus the absolute value of cash value difference of asset holdings, plus the absolute value of cash value difference of bond holdings. The cash value of asset resp. bond holdings are calculated with the equilibrium price of assets respectively bonds. The allocation error indicator can be interpreted as the average amount of cash mis-allocated.

The results of the simulated double auctions could be briefly summarized as follows. In answer to the first question, the last columns of Table 2 shows that convergence does indeed take place in at least 94% of the simulation runs. However, without ABM convergence takes place only in 40% of the simulation runs, and without BP there is no convergence at all. In answer to the second question, column 5 of Table 2 shows that the price error is at most 2.01%. Without ABM the price error can be up to 8.36%, and without BP the price error can be as high as 15.77%.
In answer to the third question, column 6 of Table 2 shows that the allocation error is at most 0.295. Without ABM the allocation error can be up to 1.10, and without BP up to 0.945. (Compare these number to the total cash values of an agent’s endowment, which are 1.716 respectively 1.749.) The allocation profiles of pessimistic and optimistic agents are sharp in equilibrium but in the simulated auctions they show a more gradual transition (see top left panel in Fig. 2). There are some residuals remaining due to an inefficiency of the double auction: Towards the end the agents around \( i = 0.8 \) hold some unpledged assets, whereas in the equilibrium allocation there are no unpledged assets. Another, smaller, inefficiency shows up in the vicinity of the \( i = 0.58 \), where some agents hold both cash and bonds simultaneously. In the equilibrium allocation no agent (or only the marginal agent \( i_1 \)) holds both cash and bonds, see the right hand panel of Fig. 1.

Regarding the fourth question, the endogenous contract selection of collateral equilibrium theory is well approximated by the final allocation of the double auction (see Table 3). Maximally 6.26% of traded bonds are of the wrong type. Without ABM up to 44.66% are of the wrong type, and without BP up to 21.49%. The number of unpledged assets is another indicator of misallocation. In collateral equilibrium there should be no unpledged assets. The proportion of unpledged assets is up to 0.87% for our standard auction mechanism, but without ABM or BP the proportion of unpledged assets can be between 40% and 70%.

In sum, the final state of our standard auction mechanism approximates collateral equilibrium reasonably well. But mechanisms without ABM or without BP lead to prices and allocations clearly different from equilibrium. These institutional features of the auction mechanism are crucial for convergence of the double auction to collateral equilibrium. Actually, the design of institutions should take into account the information, strategies, and intelligence of the agents. If agents have a high probability of taking suboptimal decisions, the institutions should provide for smooth mechanisms to reverse suboptimal trades. This requires a comprehensive view of the utility of asset and bond trades.

### 3.4. Robustness analysis of simulation results

In this Section we analyze the robustness of simulation results when changing various model assumptions. We analyze how simulation outcomes change (A) when there are only 30 instead of 1000 agents, (B) when agents differ in their initial endowment, and (C) when the degrees \( i \) of optimism are not equally spaced on \([0, 1] \). More specifically, in the robustness analysis (B) we assume that every tenth agent has a higher initial endowment in the real asset (6 instead of 1 unit) and the other agents have a lower initial endowment (4/9 units instead of 1 unit). The total amount of assets in the economy is unchanged. Initial cash endowments are not changed. In the robustness analysis (C) we assume the distance of \( i \)-values of the agent population decreases geometrically in the range \([0, 0.5] \) and increases geometrically in \([0.5, 1] \). So the \( i \)-values of the agent population are less dense in the range of extreme optimism and pessimism.

Fig. 2 shows the final allocations in the standard simulation and in the robustness analysis (A), (B), and (C). The emergence of the three groups of P-agents, M-agents, and O-agents with their distinct portfolios is clearly robust to all changes considered. The second panel of Table 4 shows that price error remains small in robustness analysis (A) but the allocation error is considerably larger than in the market with 1000 agents.

<table>
<thead>
<tr>
<th>Face values</th>
<th>Proportion of bonds issued (%)</th>
<th>Unpledged assets (%)</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Analytical equilibrium</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Final allocation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>97.56</td>
<td>1.57</td>
<td>0.87</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>93.40</td>
<td>6.26</td>
<td>0.38</td>
</tr>
<tr>
<td><strong>Final allocation in market mechanism without ABM</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>0</td>
<td>44.66</td>
<td>55.34</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>28.40</td>
<td>27.65</td>
<td>43.95</td>
</tr>
<tr>
<td><strong>Final allocation in market mechanism without BP</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>41.70</td>
<td>21.49</td>
<td>36.81</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>12.22</td>
<td>17.48</td>
<td>70.30</td>
</tr>
</tbody>
</table>
4. Limits of the double auction

The double auction with zero intelligence agents ceases to converge to the theoretical equilibrium when restriction are imposed on which agents can trade with each other. In a variation of robustness analysis (A) above we considered 30 agents who now can trade only with their next neighbors in a network specified in the left hand plot of Fig. 3. The network consists of three components where trade is not restricted. The three components are linked only via the three agents with \(i = 1/3, 2/3, 1\), who act as hubs.

The final allocation arising in this network is shown in the right hand plot of Fig. 3. Although in both cases there is the same number of agents, it is clearly different from the final allocation without restrictions on trade, see top right panel in Fig. 2.

### Table 4
Simulation results with unrestricted trade and trade restricted according to the network in Fig. 3. 30 agents with equally spaced degree of optimism, as in robustness analysis (A). The prices are averages and standard deviations (in parentheses) over the 1% final transaction prices of 50 auctions. For markets not active in the final 1% of transactions the price entries are left empty. The price error is defined on p. 11. The convergence indicator is defined on p. 11.

<table>
<thead>
<tr>
<th>Bond type</th>
<th>Prices</th>
<th>Price error (%)</th>
<th>Allocation error</th>
<th>Convergence (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Asset/cash p</td>
<td>Bond/cash q</td>
<td>Asset/bond p/q</td>
<td></td>
</tr>
<tr>
<td>Theoretical equilibrium</td>
<td>0.5</td>
<td>0.716</td>
<td>0.374</td>
<td>1.912</td>
</tr>
<tr>
<td>Final prices in unrestricted auction(A))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>–</td>
<td>–</td>
<td>1.911</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0055)</td>
<td>(0.16)</td>
</tr>
<tr>
<td>Final prices in auction on network</td>
<td></td>
<td></td>
<td>1.848</td>
<td>5.44</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.0823)</td>
<td>(3.42)</td>
</tr>
</tbody>
</table>

Fig. 3. Double auction with restricted trade. Left: Network of possible trade relations. Numbers of knots indicate the probability \(i\) assigned by the respective agent to the \(U\)-state in units of \(1/30\). Right: Final allocation after completion of trade in the network. Face value of available bonds is 0.5. The final allocation is clearly different from the final allocation without restrictions on trade, see top right panel in Fig. 2.
transaction prices is increased from 0.16% for the unrestricted auction to 3.42% for the auction on the network. This could be interpreted as arising from the emergence of four overlapping sub-markets, one for each of the three network components, and one for the three ‘hubs’ by which the components are connected. Transactions on the four sub-markets continue to the end of the auction, but prices in the submarkets differ due to the different populations of the sub-markets. The exchange between the three components via the hubs works only imperfectly, and therefore these price differences persist to the end of the auction.

One might consider the possibility that the network structure should enter in the equilibrium concept. For bipartite graphs such a bargaining equilibrium concept was developed for example by Chakraborty et al. (2009). Our graph is not bipartite due to the heterogeneous population of agents, and in our economy not just assets are traded but asset and bond markets are linked through the collateral and the budget constraints.

5. Convergence of the double auction

When only risky bonds are available, the dynamics of allocations during the auction shows four distinct stages, see Figs. 4 and 5. In Stage 1 the pessimists sell both assets and bonds, usually against cash. The optimists buy both assets and bonds against cash. The extreme pessimists and the extreme optimists are more active, since their offers have a higher probability of being matched. This stage comes to an end when the extreme optimists run out of cash.

In Stage 2 the pessimists sell assets against bonds. The optimists buy assets against both cash and bonds. This stage ends when the pessimists have no more assets to sell. They become passive since there are no more transactions possible for them. Going short on bonds is impossible because they have no assets to pledge as collateral.

In Stage 3 the more moderate pessimists, who are still in the auction, buy bonds against assets. The optimists continue to buy assets against bonds. At this stage the asset/cash and the bond/cash markets cease to be active because the pessimists,
who hold all cash, do not want to buy assets or bonds. This stage comes to an end when the extreme optimists have no more
free assets and they therefore cannot sell any more bonds.

In Stage 4 trading in the asset/bond market continues and the bond/cash market restarts. On the bond/cash market
agents in a narrowing range around some $i_1\in I$ are active. Those less optimistic than $i_1\in I$ sell bonds for cash, those more
optimistic than $i_1\in I$ buy assets for bonds. On the asset/bond market agents close to some $i_2\in I$ remain active longest. Those more
optimistic than $i_2\in I$ buy assets for bonds, those less optimistic than $i_2\in I$ sell their assets against bonds. Stage 4 comes to an end,
and this is also the end of the auction, when all agents cease to be active: The pessimists have no assets and therefore can
sell neither assets nor bonds. The intermediate agents have neither assets nor cash, and thus they cannot continue to buy
bonds. The optimists have no more cash, and they cannot sell any more bonds for lack of free assets pledgeable as collateral.

The mechanism producing price convergence in Stage 4 is quite intuitive. For example on the bond/cash market, agents
farther below $i_1\in I$ sell their bonds first because they are willing to accept a lower price, and therefore have a higher probability
of making a transaction. Agents farther above $i_1\in I$ get bonds first because they are willing to pay a higher price, and therefore
have a higher probability of making a transaction. As Stage 4 continues the agents farther below and above $i_1\in I$ leave the
bond/cash market because they have no more bonds to sell or because they have no more cash to buy bonds. Agents closer
to $i_1\in I$ then get a chance to make their transactions. In this process the range of agents making transactions concentrates more
and more around $i_1\in I$. Transaction prices converge to the price at which $i_1\in I$ is indifferent between buying and selling bonds
against cash.

This mechanism explains why transaction prices in the auction converge. The final prices are the indifference prices of
the final agents $i_1\in I$ and $i_2\in I$, which are not necessarily the marginal agents $i_1, i_2$. Whether convergence is to the equilibrium
price, is a different question. If $i_1 = i_2$, the final agents remaining are very close to the marginal agent $i_1$, who at the
equilibrium price is indifferent between buying and selling bonds for cash. Then the final prices are equal to the equilibrium
prices. But if the agents last active are very close to some $i_1\in I$ which is significantly different from the marginal $i_1$, the final
price, which equals the indifference price of $i_1\in I$, will be different from the equilibrium price. This is in fact what happens. The
final prices are slightly, but significantly different for the equilibrium prices, see Table 2. Indeed the final prices are the
indifference prices of the final agents $i_1\in I$ and $i_2\in I$, not of the marginal agents $i_1, i_2$.

Another issue is that in the simulations there remains some allocation inefficiency. One kind of inefficiency is reflected by
a rest of unpledged assets held by agents around $i_2\in I$. This rest appears in the final allocation shown in the bottom right panel
of Fig. 4. Another kind of inefficiency is reflected by the existence of a range of agents around $i_1\in I$ for whom no constraint is
binding. The agents around $i_1\in I$ hold both cash and bonds. This second inefficiency concerns a much narrower range of agents,
see the bottom right panel of Fig. 4. This is the reason why convergence is more efficient and price errors are smaller on the
bond/cash market than on the asset/bond market, see the results in Table 2 for the bond of type 0.5.

6. Conclusions

We have developed a framework for assessing whether recent advances in the general equilibrium theory of trade in
leveraged assets provide an appropriate perspective on actual market outcomes. Rather than studying field data we have
constructed an agent based double auction to experimentally investigate whether two main features of the theory, endogenous leverage and above fundamental value asset pricing, do occur in simulated markets. We investigate why or why not theory has predictive power in this environment and when and why it predicts poorly.

Our main finding is that agreement of the final state of our double auction mechanism and of the collateral equilibrium state depends crucially on the specific institutional details of exchange. More specifically, it is crucial (i) that institutions allow for easy reversibility of suboptimal trades during the auction, (ii) institutions allow for flexible coordination between debt and collateral markets, and (iii) bilateral trading is not restricted. In the double auction these details matter because agents do not make fully optimal decisions at each stage of the trading process. If the institutions of exchange support a swift and flexible reversion of suboptimal decisions and provide good coordination devices between debt and collateral markets, then an abstract analysis that builds on optimal behavior provides accurate predictions.

Our double auction provides such mechanisms with the result that in simulations with zero intelligence agents, the mechanism leads to price convergence on all markets. The final prices are close to the equilibrium prices. The limit allocations are close to equilibrium allocations, although some inefficiencies remain. The bond type selection (endogenous leverage) predicted by collateral equilibrium theory can also be observed in the double auction. Both the trade reversibility and the coordination devices are key elements of our market mechanism and are crucial for convergence to equilibrium. Without the asset/bond markets (ABM), or without pledgeability of bonds (BP), convergence to equilibrium prices, convergence to equilibrium allocations, and endogenous leverage (bond type selection) breaks down.

These two institutional features of the auction are therefore key: they are necessary for the abstract theory to provide an appropriate perspective on the outcomes of the particular market mechanism. The result shows that institutions have to take into account the information, behavior and intelligence of the agents. For agents with different information processing capacities, different market institutions could work.

The agreement of the final state of the double auction and of the collateral equilibrium state is robust under variation of the number of agents, of their initial endowment, and of the distribution of their preferences. But agreement ends when trade restrictions are introduced. When trading is restricted to next neighbors in a network, the final allocation reflects the structure of the network and is clearly different from the equilibrium allocation.

The findings could be interpreted as hinting at some key elements that make collateral theory work in practice and helps to gauge the scope of its applicability. Our findings suggest that the field where the theory is perhaps most appropriately applied are where both debt and collateral is traded competitively and products as well as exchange procedures are fairly standardized. The theory is likely to provide an appropriate perspective on haircut and asset prices in these markets. Whether and under which circumstances the theory also provides an appropriate perspective on loan to value ratios and housing prices need further research. While at the level of the theory no changes (except that the collateral good also provides direct utility) have to be made to analyze mortgage lending it is not so clear whether and under which circumstances the abstraction from the institutional details of exchange are innocuous in this case. In particular, trade restrictions, be they de iure or de facto, seem to have a major impact on prices and allocations, with detrimental welfare effects. We leave this question for future research.

Disclaimer

Any opinions, findings, and conclusions or recommendations express in this material are those of the authors and do not necessarily reflect the views of OeNB.

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Appendix A. Some details for the benchmark example

The main attraction of the benchmark example is that in this framework the computation of equilibrium is relatively simple. In this appendix we discuss two technical points: (i) We show some details of the equilibrium calculation. (ii) We prove our claims for the case where the financial market structure does not allow for riskfree debt. This is a case not discussed in Geanakoplos (2010).

A.1. Collateral equilibrium with riskless debt

Assume first that only one bond, the riskfree bond with face value $V=0.2$, is available. Denote the bond price of the real asset by $p$ and the bond price by $q$. The consumption plan of agent $i$ is denoted by $x^i = (x^i_0,x^i_2,x^i_3)$, where $x^i_0$ is consumption today and $x^i_2$ and $x^i_3$ is consumption in state $U$ and $D$ tomorrow. Let agent $i$’s planned holdings of the real asset be denoted by $y^i$ and the bond holdings by $z^i_+ \text{ if long (lending)}$ and $z^i_- \text{ if short (borrowing)}$. We use the same parametrization as
Geanakoplos (2010), where the real asset pays 1 in the U state and 0.2 in the D state and every agent has an endowment of 1 today of the consumption good and one real asset. There are no endowments tomorrow.

With these conventions we can write the choice set of an agent $i \in [0, 1]$ as

$$E(p, q) = \begin{cases} 
\mathbf{x}' \in \mathbb{R}^+ \mid \begin{align*}
0.2 q \mathbf{z}' &\leq y' + 0.2 \mathbf{z}'_+
\end{align*} \\
\mathbf{z}' \leq y' \\
(z_0', \mathbf{z}_0', \mathbf{z}'_+) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+
\end{cases}$$

(1)

Since preferences are linear, depending on their degree of optimism, agents either want to buy or sell the real asset. This decision depends on whether the asset price $p$ is larger of lower than the expected value of the dividends provided by the real asset. The agent with the value $i^* \in [0, 1]$ such that $i^*$ solves the equation:

$$p = i 1 + (1 - i) 0.2$$

is indifferent between buying and selling the real asset. This solution is given by

$$i^* = \frac{p - 0.2}{0.8}$$

Agents with $i > i^*$ are optimistic and would like to sell bonds in order to buy more of the real asset, since they believe it is worth more than it costs in the market. Since preferences are increasing in consumption the budget is fully exhausted. Since an optimist borrows as much as he can, his collateral constraint must be binding and thus $z'_1 = y'$. He does not want to lend, thus $z'_0 = 0$. Also by the linearity of preferences for the optimist only the sum of total consumption matters. Since he is optimistic about the $U$ state in an optimal plan he puts all of his contemporaneous resources into purchasing real assets. Therefore at an optimal plan, for him $x_0' = 0$. Thus each O-agent holds $y' = (1 + p)/(p - q)$ units of the real asset and thus has sold the maximum possible amount $z'_0 = (1 + p)/(p - q)$ of bonds.

Pessimistic agents, agents with $i \leq i^*$ want to sell the asset and buy only the consumption good or the bond. In equilibrium the pessimistic agents must be indifferent between buying the bond and the consumption good. The only way this can be the case is, if the bond price is equal to

$$q = V$$

In equilibrium the aggregate demand for the real asset must equal the aggregate supply of 1. Thus in equilibrium the condition:

$$\int_{i^*}^1 (1 + p)/(p - q) \; di = 1$$

must hold. This is equivalent to the equation:

$$p = \frac{1 + q - i^*}{i^*}$$

In summary we can characterize the equilibrium by the solution to the system:

$$i^* = \frac{p - 0.2}{0.8}$$

(2)

$$p = \frac{1 + q - i^*}{i^*}$$

(3)

$$q = 0.2$$

Geanakoplos (2010) shows that when additional risky bonds are available this equilibrium remains unchanged. While the marginal buyer and prices of the real asset and the riskless bond remain unchanged, the risky bonds are priced but not traded. If the agent would for instance want to borrow using the bond with face value of 0.5 he would have to pay

$$q = i^* 0.5 + (1 - i^*) 0.2$$

where $i^*$ is the solution to (2). But at these prices pessimists would not want to lend. Also optimists would not want to borrow at these terms because it requires from them to pay the same amount in the D-state but more than with a riskless loan in the U state about which he is very sure that it will occur. Thus there is contract selection.

The collateral premium can be seen from comparing the asset price in the absence of bonds (or with bond face value zero) to the asset price in the presence of bonds of non-vanishing face value. Without bonds equilibrium is determined by the equations:

$$i^* = \frac{p - 0.2}{0.8}$$

(5)
The price of the real asset in this case is only \( p = 0.68 \) instead of \( p = 0.75 \). The difference between these two prices comes from the fact that asset purchase can be leveraged. These additional resources in the hands of optimists allow a price rise of the real asset over and above its fundamental value.

### A.2. Collateral equilibrium without riskless debt

We additionally describe the equilibrium in situations where only risky bonds are available. Denote by \( V \) the lowest face value of the available bonds, \( V > 0 \). Geanakoplos (2010) excludes this case by assumption. In this case there are three (rather than two as in the case of a risk free bond) groups of agents: P-agents have \( i \in [0, i_1] \), the medium agents or M-agents have \( i \in [i_1, i_2] \), and O-agents have \( i \in [i_2, 1] \). In equilibrium each P-agent holds \( 1 + p \) units of the consumption good and nothing else, each M-agent holds \( (1 + p)/q \) bonds and nothing else and each O-agent holds \( (1 + p)/(p - q) \) units of the real asset and has sold the maximum possible number \( (1 + p)/(p - q) \) of collateralized bonds. The agent \( i_1 \) is indifferent between holding just the consumption good (as the P-agents) or holding just bonds. The agent \( i_2 \) is indifferent between holding only bonds (as the M-agents), or holding only assets purchased in exchange for the consumption good, or holding assets purchased in exchange for the consumption good and the proceedings from selling the maximum possible number of bonds (as the O-agents).

**Proposition 1** (Prices of assets financed by risky collateralized bonds). In case only one bond type is available, and this bond is risky \( (V > 0 \), the asset price \( p \), the bond price \( q \), and the allocation parameters \( i_1, i_2 \) are given by the solution of the following equations:

\[
\begin{align*}
  i_1 &= \frac{q - 0.2}{V - 0.2} \\
  i_2 &= \frac{0.2(p - q)}{0.8q - (V - 0.2)p} \\
  p &= \frac{1}{i_1} - 1 \\
  q &= p \frac{i_2 - i_1}{1 - i_1}
\end{align*}
\]

**Proof.** Agents can invest their endowments in four different ways. They can sell their asset endowment and just keep \( 1 + p \) units of cash (P-agents), they can sell their asset endowment and use their \( 1 + p \) units of cash to buy \( (1 + p)/q \) units of bonds at unit price \( q \). These are the M-agents who buy bonds with all their cash. Agents could also use their cash endowment to buy \( 1/p \) units of the asset, leaving them with \( (1 + p)/p \) units of the asset and nothing else. Call this type of agent R-agent. It will turn out that for no agent this cash-financed asset investment is strictly preferable to all other investments. Only agent \( i_2 \) is a possible R-agent, but he is indifferent between this and other investment opportunities. The fourth kind of investment, preferred by the O-agents, is to finance asset purchases by a combination of cash from the endowment and bond sales proceeds. Other portfolios are possible but not optimal for any agent. For example, no agent would hold cash and at the same time sell bonds because the expected amount to be paid back on the bond is more than cash he receives as bond. And no agent would strictly prefer to buy a bond and at the same time sell bonds. Furthermore, portfolios diversifying into more than one of the four above investment opportunities will not be strictly preferable to any agent. Since there is only one source of risk (whether the up or the down state is realized) there are no diversification benefits to be gained.

Let us now derive the allocation parameters \( i_1, i_2 \). Agent \( i_1 \) is indifferent between holding \( 1 + p \) units of the consumption good or holding \( (1 + p)/q \) bonds: \( 1 + p = (V_i + 0.2(1 - i_1))(1 + p)/q \). Solving for \( i_1 \) we get (7).

Agent \( i \), being the marginal agent between type M and type R, is indifferent between holding \( (1 + p)/q \) bonds and holding \( (1 + p)/p \) units of cash-financed assets: \( (V_i + 0.2(1 - i))(1 + p)/q = (i + 0.2(1 - i))(1 + p)/p \). Solving this for \( i \) we get

\[
\hat{i} = \frac{0.2(p - q)}{0.8q - (V - 0.2)p}
\]

Denote by \( \hat{i} \) the marginal agent who is both an R-type and an O-type. He is indifferent between holding just cash-financed assets and holding assets financed by both cash and the maximal amount of bonds available to him:

\[
\frac{1 + p}{p} \hat{i} + 0.2 \left( 1 - \hat{i} \right) = a_0 \left( \hat{i} + 0.2 \left( 1 - \hat{i} \right) \right) + b_0 \left( V \hat{i} + 0.2 \left( 1 - \hat{i} \right) \right),
\]

where \( a_0 \) is the number of assets held by the O-agents, \( b_0 < 0 \) is the number of bonds held by the O-agents. O-agents would like to sell as many bonds as possible to raise as much cash as possible to buy assets. Their only restriction is the collateral constraint, which prevents them from selling more bonds than they have assets. Thus \( a_0 = -b_0 \). Since \( a_0 = (1 - b_0 q)/p + 1, \)
the number of assets held by the O-agents equals

\[ a_0 = \frac{p+1}{p-q} \]

Entering this into (12) a few lines of algebra establish

\[ \iota = \frac{0.2(p-q)}{0.8q - (V - 0.2)p} \]

(13)

Therefore \( \iota = \iota \). Denoting this number by \( \iota_2 \) we have (8). The group of R-agents collapses to the single agent \( \iota_2 \), who is at the same time an M-agent, an R-agent, and an O-agent. He is indifferent between holding just bonds, or holding just cash-financed assets, or holding assets financed by cash and by selling as many bonds as allowed by the collateral constraint.

Now let us finally consider markets where two or more bond types are available simultaneously. From the theory of collateral equilibrium we know that an equilibrium exists. Under the restriction that the bond type \( V = 0.2 \) is available, which raises the highest amount of cash among the risk free bonds, Fostel and Geanakoplos (2011) already showed that only this bond type will be traded. Other bond types will not be traded, but their value is determined by equilibrium conditions. (If those bonds had a lower value, M-agents would prefer to buy them, but O-agents would prefer to sell the \( V = 0.2 \) bond, so that we cannot have an equilibrium. If those bonds had a higher value, O-agents would prefer to sell them, but M-agents would prefer to buy the \( V = 0.2 \) bond, so that we cannot have an equilibrium either.) This bond type selection is what (Geanakoplos, 2010) called endogenous determination of leverage in equilibrium. Unique bond type selection is not a universal property of collateral equilibrium. Fostel and Geanakoplos (2011) show that it fails when either the asset, apart from paying a consumption good dividend, provides some utility in itself (like housing), or if three or more states are possible, rather than just the two states up and down. But bond type selection does not depend on the infinite number of agents involved in this particular model. It also happens when the number of agents is finite.

Now how does equilibrium look like when there are several risky bonds but the risk free bond offering highest leverage (\( V = 0.2 \)) is not available? To begin with, assume that there are two bond types \( V_1, V_2 \).

**Proposition 2** (Contract selection for risky collateralized bonds). If both bond types are risky \((0.2 < V_1 < V_2 < 1)\), only the less risky bond type \( V_1 \) will be traded. If three or more risky bond types are available, the least risky will be traded.

**Proof.** We use \( V \) to denote the face values \( V_i \) or \( V_j \) and \( q_j \) to denote the price of the bond with face values \( V \). The choice of bond types is determined by the O-agents, who sell bonds, and by the M-agents, who buy bonds. For agent \( i \), the expected utility of an O-type portfolio, which holds the maximum amount of assets, financed by a combination of cash and the maximum amount of type \( j \) bonds equals

\[ U_{o}(i) = \begin{cases} 
(1 + p)(1 - V)/(p - q) & \text{if } V > 0.2, \\
(1 + p)(0.2 - V)/(p - q) + i(1 + p)0.8/(p - q) & \text{if } V \leq 0.2.
\end{cases} \]

(14)

The expected utility of an M-type portfolio, consisting just of bonds of type \( j \), for agent \( i \), equals

\[ U_{m}(i) = \begin{cases} 
(1 + p)0.2/q + i(1 + p)(V - 0.2)/q & \text{if } V > 0.2, \\
1 + p & \text{if } V \leq 0.2.
\end{cases} \]

These equations result from the expected payoff equations, when we use the collateral constraint \( a_0 = b_0 \) and the expression \( a_0 = (1 + p)/(p - q) \).

Eq. (14) implies that all agents, irrespective of their \( i \), would prefer the same bond type to finance asset purchases, namely the \( V_1 \) with a higher value of \((1 + p)(1 - V)/(p - q)\). Bonds with a lower value of \((1 + p)(1 - V)/(p - q)\) will not be demanded. This implies that contract selection takes place.

To prove that it is bond type \( V_2 \) which will not be traded, we show that the negation (bond type \( V_2 \) is traded) leads to a contradiction. Since contract selection takes place, the assumption that bond \( V_2 \) is traded implies that bond \( V_1 \) is not traded. This, by (14), implies

\[ \frac{1 - V_2^2}{p - q_2} > \frac{1 - V_1}{p - q_1}. \]

(15)

Since not both \( V_1, V_2 \) are risky free, we know that there are three groups of agents. So there are also M-agents, who prefer to buy bonds (of either type) to either holding cash or holding assets (financed either by cash or by a combination of cash and proceeds from selling bonds of either type). The payoff of a \( V_2 \)-bond is equal to the payoff of a \( V_1 \)-bond and at the same time holding \( V_2^2 - V_1^2 \) units of the state \( U \) Arrow security (which pays 1 in state \( U \) and nothing otherwise). The price of the state \( U \) Arrow security equals \((p - q_2)/(1 - V_2)\) since the O-agents need to hold \( 1/(1 - V_2) \) units of asset and sell \( 1/(1 - V_2) \) units of \( V_2 \)-bonds to replicate the payoff of the state \( U \) arrow security. Thus

\[ q_2 = q_1 + (V_2 - V_1) \frac{p - q_2}{1 - V_2^2} \]
which implies

\[
\frac{V^2 - V^1}{q_2 - q_1} = \frac{1 - V^2}{p - q_2} - \frac{1 - V^1}{p - q_1}
\]

and the fact that \(a/b = c/d\) implies \((a+c)/(b+d) = c/d\) imply

\[
\frac{1 - V^2}{p - q_2} > \frac{1 - V^1}{p - q_1} = \frac{(V^2 - V^1) + (1 - V^2)}{(q_2 - q_1) + (p - q_2)} = \frac{1 - V^2}{p - q_2}
\]

which is a contradiction. \(\square\)

**Proposition 3** (No interference of non-traded bonds). The asset price and the price of the traded bond are the same as if the non-traded bond type were not available, see (7)–(10). The price \(q_2\) of a non-traded risky bond is determined by the condition

\[
0.8q_2 - (V^2 - 0.2)p = 0.8q_1 - (V^1 - 0.2)p
\]

where \(q_1\) is the price of the traded risky bond.

**Proof.** Let us consider the case where two risky bonds \(0.2 < V^1 < V^2\) are available. From (11) and (13) we know that, at a \(V^1\)-bond price of \(q_2\), the agent \(i = i' = (0.2p - q_1)/(0.8q_1 - (V^1 - 0.2)p)\) is indifferent between holding only \(V^1\)-bonds, or holding only assets financed by the sale of \(V^1\)-bonds at price \(q_1\). Fixing \(p\) and \(V^2\), the agent with \(i\) equal to

\[
0.2(p - q_1) - 0.8q_2 - (V^2 - 0.2)p
\]

is indifferent between holding only \(V^2\)-bonds, or holding only assets financed by selling the maximum number of \(V^2\)-bonds at price \(q_1\). Now chose the \(V^1\)-bond price \(q_2\) such that

\[
0.2(p - q_1) = 0.2(p - q)\frac{0.8q_1 - (V^1 - 0.2)p}{0.8q_2 - (V^2 - 0.2)p}
\]

This is (17). Denote by \(q_2^*\) the value of \(q_2\) solving this equation. If the \(V^2\)-bond price equals \(q_2^*\), the agent \(i = 0.2(p - q_1)/(0.8q_2 - (V^2 - 0.2)p)\) is indifferent between four portfolios: holding only \(V^1\)-bonds, holding only assets financed by \(V^1\)-bonds sold at price \(q_1\), holding only \(V^2\)-bonds, holding only assets financed by \(V^2\)-bonds sold at price \(q_2^*\).

For this \(V^2\)-bond price \(q_2^*\), the equilibrium looks the same as if no \(V^2\)-bonds were available: Agents \(1 \leq i_{t}(V^1)\) are \(P\)-agents and hold \(1 + p\) units of cash; agents with \(i_{t}(V^1) < i < i_{t}(V^1)\) are \(M\)-agents and hold only \(V^1\)-bonds; agents with \(i_{t}(V^1) < i\) are \(O\)-agents and hold only assets financed by \(V^1\)-bonds sold at price \(q_1\). If the price of \(V^2\) bonds were higher than \(q_2^*\) the \(O\)-agents would prefer to finance their asset purchases by selling \(V^2\)-bonds. This cannot be an equilibrium since all \(M\)-agents prefer to buy \(V^1\)-bonds. If the price of \(V^2\) bonds were lower than \(q_2^*\), the more pessimistic \(M\)-agents would prefer to buy \(V^2\)-bonds, whereas the more pessimistic \(M\)-agents still would prefer to buy the \(V^1\)-bonds. This is not an equilibrium, because all \(O\)-agents prefer to finance their asset purchases by selling \(V^1\)-bonds. \(\square\)

References


